

Quantum Painlevé-Calogero Correspondence

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Abstract

The Painlevé-Calogero correspondence is extended to auxiliary linear problems associated with Painlevé equations. The linear problems are represented in a new form which has a suggestive interpretation as a “quantized” version of the Painlevé-Calogero correspondence. Namely, the linear problem responsible for the time evolution is brought into the form of non-stationary Schrödinger equation in imaginary time, $\partial_t \psi = (\frac{1}{2} \partial_x^2 + V(x, t)) \psi$, whose Hamiltonian is a natural quantization of the classical Calogero-like Hamiltonian $H = \frac{1}{2} p^2 + V(x, t)$ for the corresponding Painlevé equation.

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1 Introduction

The famous six nonlinear ordinary second-order differential equations discovered by P.Painlevé, R.Fuchs and B.Gambier [1, 2, 3] in the beginning of the XX century are nowadays known as the Painlevé equations I–VI (P_I – P_{VI}). Since that time they were extensively studied and they still remain to be among the most important and most interesting differential equations in mathematics and mathematical physics [4, 5]. Their applications include self-similar reductions of non-linear integrable partial differential equations [6], correlation functions of integrable models [7, 8], quantum gravity and string theory [9], topological field theories [13], 2D polymers [10], random matrices [11, 12] and stochastic growth processes [14], to mention only few applications and few references.

The idea to associate a system of *linear* differential equations with each Painlevé equation goes back to the seminal work by R.Fuchs [2]. In fact the theory of Painlevé equations is intrinsically related to the monodromy properties of linear ordinary differential equations with rational coefficients. Remarkably, the equations from the Painlevé list describe monodromy preserving deformations of linear differential equations with essential singularities. The classical references on the subject are [15, 16]. The monodromy approach was further developed by H.Flaschka and A.Newell [6] and by M.Jimbo, T.Miwa and K.Ueno in the series of works [17, 18, 19], see also book [20]. At present different types of linear problems (scalar [2, 15], 2×2 -matrix [18] or 3×3 -matrix [21]) are known to be associated with Painlevé equations.

The Hamiltonian theory of the Painlevé equations is dated back to the work [22] (for the modern developments and the extension to general Schlesinger systems see [23]). It turns out that all the six equations have a Hamiltonian structure with time-dependent Hamiltonian functions which are polynomials in the dependent variable (the coordinate) and suitably chosen conjugate momentum. They are referred to as Okamoto’s Hamiltonians [24]. However, the Okamoto’s Hamiltonians for P_{II} – P_{VI} equations are of a more complicated form than just momentum squared plus potential. This makes a direct interpretation of Painlevé equations as classical mechanical systems (a point-like particle on the line moving in a time-dependent potential) problematic. Nevertheless, such an interpretation appears to be possible after a non-trivial canonical transformation which accomplishes the *Painlevé-Calogero correspondence*.

The phenomenon known in the literature as the (classical) Painlevé-Calogero correspondence [25] consists in the possibility to represent, by means of explicitly known transformations of the dependent and independent variables, all the six Painlevé equations as non-autonomous Hamiltonian systems

$$\partial_t x = \frac{\partial H}{\partial p}, \quad \partial_t p = -\frac{\partial H}{\partial x}$$

with the standard one-particle Hamiltonian of the canonical form $H = p^2/2 + V(x, t)$ for some potential $V(x, t)$ which explicitly depends on time t . In the case of P_{VI} this Hamiltonian system resembles the elliptic Calogero model with 2 particles in the center of mass coordinates, whence the name Painlevé-Calogero correspondence. (To be more precise, the P_{VI} equation is a non-autonomous version of a special rank-one case of the Inozemtsev’s extension [26] of the elliptic Calogero model.) For the P_{VI} equation this remarkable observation was made by Yu.Manin [27] who revived the almost forgotten

work by Painlevé himself [28]. Later, K. Takasaki [29] extended this result to the other equations from the Painlevé list. In principle, this extension can be achieved by a special degeneration process from P_{VI} to the lower members of the Painlevé family. Although the resulting Hamiltonian systems hardly resemble any Calogero-like models, the name “Painlevé-Calogero correspondence” has been extended to these cases as well. This also suggests generalizations to higher rank systems which were studied in [29].

The explicit form of the canonical transformations from the Okamoto’s Hamiltonian systems to Calogero-like ones was found in [29]. Here we need only the coordinate part of this transformation which is described by the following theorem.

Theorem 1 [29]. *For any of the six equations from the Painlevé list written for a variable $y(T)$ there exists a change of variables $(y, T) \rightarrow (u, t)$ of the form $y = y(u, t)$, $T = T(t)$ that maps the Painlevé equation to a second-order differential equation of the form*

$$\ddot{u} = -\partial_u V(u, t) \quad (1.1)$$

which is equivalent to a non-autonomous Hamiltonian system $\dot{u} = \partial H(p, u, t)/\partial p$, $\dot{p} = -\partial H(p, u, t)/\partial u$ with the Hamiltonian

$$H(p, u, t) = \frac{p^2}{2} + V(u, t) \quad (1.2)$$

where $V(u, t)$ is a time-dependent potential written in terms of rational, hyperbolic or elliptic functions.

This statement was proved in [29] by giving explicit formulas for the corresponding changes of variables (see the table below). We call (1.1) the *Calogero form* of the Painlevé equation.

The aim of this paper is to extend the Painlevé-Calogero correspondence to the linear problems associated with the Painlevé equations. In fact we suggest a new form of the linear problems which allows us to interpret it as a “quantized” version of the Painlevé-Calogero correspondence. In other words, linearization, i.e., going to the associated linear problems, appears to be equivalent to quantization of the Painlevé equations regarded as classical mechanical systems.

The starting point is a system of two first-order linear partial differential equations (PDE) in two variables for a 2-component vector-function $(\psi_1, \psi_2)^t$ of the form presented, for example, in [18]. The two variables are the spectral parameter and the deformation parameter. As is well known, compatibility of the system is equivalent to the zero curvature condition for the connection represented by 2×2 matrices depending on the two variables. The next step is the change of the dependent and independent variables that leads to the Calogero-like form of the Painlevé equations, supplemented by a suitable change of the spectral parameter (polynomial for P_I , P_{II} , P_{IV} , exponential for P_{III} , hyperbolic for P_V and elliptic for P_{VI}). At this step the spectral parameter and the deformation parameter acquire the meaning of the coordinate and time variables for a non-autonomous dynamical system with one degree of freedom. After an additional diagonal gauge transformation of a special form, the linear problems transformed in this way should be rewritten as a pair of two compatible linear PDE’s for a scalar ψ -function $\psi = \psi_1$ (the first component of the vector function). One of them is an ordinary second-order differential equation

with coefficients explicitly depending on time and on the dependent variable. After a simple transformation of the ψ -function the term with the first order derivative cancels, and one obtains a stationary Schrödinger equation with a potential function which depends on time in both explicit and implicit ways, with the implicit dependence coming from the dependent variable. The isomonodromy problem for this equation, i.e., time-dependent deformation of the potential preserving the monodromy of solutions, is known to be equivalent to the Painlevé equation.

The key new element introduced in this paper is the second equation of the pair, the one describing the time evolution. We show that for all the six Painlevé equations (and any values of the standard parameters $\alpha, \beta, \gamma, \delta$ involved) it can be represented in the form of the *non-stationary Schrödinger equation* in imaginary time,

$$\partial_t \Psi = \hat{H} \Psi,$$

whose Hamiltonian is the standard 1D Schrödinger operator $\hat{H} = \frac{1}{2} \partial_x^2 + V(x, t)$ which is a *natural quantization of the classical Calogero-like Hamiltonian* associated with the Painlevé equation at hand (to be more precise, for P_{VI} the parameters $\alpha, \beta, \gamma, \delta$ in the quantized Hamiltonian appear to be shifted by “quantum corrections” $\pm \frac{1}{8}$, similar shifts of some of the parameters take place also for P_V and P_{IV}). Therein lies the quantum Painlevé-Calogero correspondence, or a classical-quantum correspondence for the Painlevé equations. Indeed, on the Calogero side, one now has a *quantum* Calogero-like or Inozemtsev model in a non-stationary state described by the wave function Ψ which differs from ψ by a coordinate-independent factor. On the Painlevé side, this Ψ -function is a common solution to the linear problems associated with the Painlevé equation. Solutions of the Painlevé equation itself can be extracted from the asymptotic behavior of the Ψ -function near singular points. The main results of this work are summarized in the following “quantum” version of Theorem 1:

Theorem 2. *For any of the six equations from the Painlevé list written in the Calogero form (1.1) as classical Hamiltonian systems with time-dependent Hamiltonians $H(p, u, t)$ (1.2) there exists a pair of compatible linear problems*

$$\begin{cases} \partial_x \Psi = \mathbf{U}(x, t, u, \dot{u}, \{c_k\}) \Psi \\ \partial_t \Psi = \mathbf{V}(x, t, u, \dot{u}, \{c_k\}) \Psi \end{cases}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (1.3)$$

where \mathbf{U} and \mathbf{V} are sl_2 -valued functions, x is a spectral parameter, t is the time variable and $\{c_k\} = \{\alpha, \beta, \gamma, \delta\}$ is the set of parameters involved in the Painlevé equation, such that

1) *The zero curvature condition*

$$\partial_t \mathbf{U} - \partial_x \mathbf{V} + [\mathbf{U}, \mathbf{V}] = 0 \quad (1.4)$$

is equivalent to the Painlevé equation (1.1) for the variable u defined as any (simple) zero of the right upper element of the matrix $\mathbf{U}(x, t)$ in the spectral parameter: $\mathbf{U}_{12}(u, t) = 0$;

2) The function $\Psi = e^{\int^t H(u, u, t') dt'} \psi_1$ where ψ_1 is the first component of Ψ satisfies the non-stationary Schrödinger equation in imaginary time

$$\partial_t \Psi = \left(\frac{1}{2} \partial_x^2 + \tilde{V}(x, t) \right) \Psi \quad (1.5)$$

with the potential

$$\tilde{V}(x, t) = V(x, t, \{\tilde{c}_k\}) = \frac{1}{2} [\det(\mathbf{U}) - \partial_x \mathbf{U}_{11} + 2\mathbf{V}_{11}]$$

which coincides with the classical potential $V(x, t) = V(x, t, \{c_k\})$ up to possible shifts of the parameters $\{c_k\}$:

$$(\tilde{\alpha}, \tilde{\beta}) = (\alpha, \beta + \frac{1}{2}) \quad \text{for } P_{IV},$$

$$(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = (\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma, \delta) \quad \text{for } P_V$$

$$(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = (\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma - \frac{1}{8}, \delta + \frac{1}{8}) \quad \text{for } P_{VI}.$$

For reader's convenience we collect the changes of variables from the original y, T to u, t required for passing to the Calogero form and the corresponding change of the spectral parameter from rational one, X , to x , in the following table:

Equation	$y(u, t)$	$T(t)$	$X(x, t)$	$\mathbf{U}_{12}(x, t)$
P_I	u	t	x	$x - u$
P_{II}	u	t	x	$x - u$
P_{IV}	u^2	t	x^2	$x^2 - u^2$
P_{III}	e^{2u}	e^t	e^{2x}	$2e^{t/2} \sinh(x - u)$
P_V	$\coth^2 u$	e^{2t}	$\cosh^2 x$	$2e^t \sinh(x - u) \sinh(x + u)$
P_{VI}	$\frac{\wp(u) - \wp(\omega_1)}{\wp(\omega_2) - \wp(\omega_1)}$	$\frac{\wp(\omega_3) - \wp(\omega_1)}{\wp(\omega_2) - \wp(\omega_1)}$	$\frac{\wp(x) - \wp(\omega_1)}{\wp(\omega_2) - \wp(\omega_1)}$	$\vartheta_1(x - u) \vartheta_1(x + u) h(u, t)$

In the last column the right upper element of the matrix $\mathbf{U}(x, t)$ is given. One can see that in all cases u is indeed a simple zero of $\mathbf{U}_{12}(x, t)$. The function $h(u, t)$ is some function of u, t only to be specified in Section 8. The Weierstrass \wp -function $\wp(z) = \wp(z|1, \tau)$ and the Jacobi theta-function $\vartheta_1(x) = \vartheta_1(x|\tau)$ in the last line of the table depend on t in a non-trivial way through the second period $\tau = 2\pi it$. The half-periods are defined as $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{1}{2}(1 + \tau)$, $\omega_3 = \frac{1}{2}\tau$.

When this work was completed, we were informed by B.Suleimanov that he realized the role of non-stationary Schrödinger-like equation in linear problems for Painlevé equations back in 1994 and obtained similar results [30]. In distinction to our approach, he starts with the scalar linear problems of the Fuchs-Garnier type with rational spectral parameter [2, 15] and shows that their compatibility implies yet another linear equation for the same wave function, which is of the non-stationary Schrödinger form, with quantum

Hamiltonian being a quantization of the corresponding Okamoto's Hamiltonian. The precise connection between the two approaches deserves further elucidation.

The presentation is organized in such a way that each Painlevé equation is discussed in a separate section, in the order of increasing complexity, from P_I to P_{VI} (Sections 3–8). We tried to make each section self-contained, so that they could be read independently of each other. However, each section contains references to Section 2, where the general construction is outlined. Note that in our list P_{IV} stands before P_{III} because in a certain sense the complexity of the latter exceeds that of the former. This is due to the fact that the P_I , P_{II} and P_{IV} equations need rational parametrization to be represented in the Calogero-like form while P_{III} and P_V require exponential and hyperbolic parametrizations for that purpose. The highest member, P_{VI} , is the most complicated object. It requires parametrization in terms of elliptic functions. One can see that the calculations which are necessary to prove Theorem 2 and to verify the classical-quantum correspondence, being really short and transparent for P_I , become very long and tedious for P_{VI} . In the case of P_{VI} (and to some extent of P_V), the situation is aggravated by the fact that neither the change of the spectral parameter nor the gauge transformation are known from the very beginning and should be either guessed or found by solving a differential equation. The three appendices are all related to the P_{VI} equation. In Appendix A some details of explicit verification of the zero curvature condition are given. Appendix B contains the necessary information on theta-functions and elliptic functions. In Appendix C the special diagonal gauge transformation together with the change of the spectral parameter for the linear problems for the P_{VI} equation is derived.

2 The general scheme

2.1 Linear problems and compatibility conditions

As is known, any Painlevé equation I-VI can be represented as the compatibility condition for a pair of linear problems depending on a spectral parameter. We need the linear problems such that they lead directly to the Painlevé equations in the Calogero form. They can be obtained from the linear problems with rational spectral parameter by a proper change of variables. The existence of such a change of variables will be proved separately for each equation P_I - P_{VI} by an explicit calculation. Now suppose that we are given with such a pair of linear problems:

$$\begin{cases} \partial_x \Psi = U(x, t) \Psi \\ \partial_t \Psi = V(x, t) \Psi \end{cases}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (2.1)$$

where the 2×2 matrices U, V explicitly depend on the spectral parameter x (which in our approach has the meaning of coordinate), on the deformation parameter t (which in our approach has the meaning of time) and contain an unknown functions of t to be constrained by the condition that the two equations have a family of common solutions. This function is going to satisfy one of the six Painlevé equations (in the Calogero form). In fact the latter is equivalent to the compatibility of the linear problems expressed as

the zero curvature equation (integrability condition)

$$\partial_x \mathbf{V} - \partial_t \mathbf{U} + [\mathbf{V}, \mathbf{U}] = 0. \quad (2.2)$$

Set

$$\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Our matrices \mathbf{U}, \mathbf{V} will be always traceless, i.e., $a + d = 0$, $A + D = 0$. In this notation, the zero curvature equation yields:

$$\begin{cases} a_t - A_x + bC - cB = 0 \\ b_t - B_x + 2aB - 2bA = 0 \\ c_t - C_x + 2cA - 2aC = 0. \end{cases} \quad (2.3)$$

Here and below a_t , A_x , etc mean partial derivatives with respect to t, x . To avoid a misunderstanding, we emphasize that the time variable t enters the matrix elements in two ways: explicit and implicit. The latter means the time dependence through the unknown functions of t (dependent variables). The notation a_t , etc implies the full time differentiating which takes into account the time dependence of the both types.

The function that satisfies the Painlevé equation in the Calogero form will be denoted by $u = u(t)$. It can be defined as zero of the right upper element of the matrix $\mathbf{U}(x, t)$ as a function of the spectral parameter x : $b(u) = 0$. We will see that this zero is always of the first order and different possible choices (in the case when the function $b(x)$ has more than one zero in a suitably chosen fundamental domain) lead to the same equation.

It is important that the matrix functions $\mathbf{U}(x, t), \mathbf{V}(x, t)$ have poles in x at the points which may depend on time *but not through the dependent variable u* . In fact for $P_I - P_V$ equations they are time independent while for the P_{VI} equation two poles are fixed and other two linearly depend on the time variable.

In what follows we will choose the matrices \mathbf{U}, \mathbf{V} such that

$$b_x = 2B. \quad (2.4)$$

(the meaning and advantages of this condition will be clear later). Given any two matrix functions \mathbf{U}, \mathbf{V} , this equality can be always attained by means of a suitable diagonal gauge transformation of the linear system (2.1) (see below). In principle, one can then exclude A and C from the zero curvature equations (2.3) and obtain a functional relation for a, b and c but we will not follow this route here. Let us only mention, for future reference, that if the zero curvature equation and the condition (2.4) are imposed, then A is expressed through a and b as follows:

$$2A = \frac{b_t + ab_x}{b} - \frac{b_{xx}}{2b}. \quad (2.5)$$

The system (2.1) admits gauge transformations $\tilde{\Psi} = \Omega \Psi$ with a matrix Ω which can depend on x, t . The gauge transformed system has the same form

$$\begin{cases} \partial_x \tilde{\Psi} = \tilde{\mathbf{U}}(x, t) \tilde{\Psi} \\ \partial_t \tilde{\Psi} = \tilde{\mathbf{V}}(x, t) \tilde{\Psi} \end{cases} \quad (2.6)$$

with

$$\tilde{\mathbf{U}} = \Omega^{-1} \mathbf{U} \Omega - \Omega^{-1} \partial_x \Omega, \quad \tilde{\mathbf{V}} = \Omega^{-1} \mathbf{V} \Omega - \Omega^{-1} \partial_t \Omega. \quad (2.7)$$

In the next sections this transformation will be applied in the opposite direction, from matrices $\tilde{\mathbf{U}}, \tilde{\mathbf{V}}$ obtained at an intermediate stage of calculations to matrices \mathbf{U}, \mathbf{V} in the final form. This is equivalent to applying the inverse transformation. In particular, let

$$\Omega = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad (2.8)$$

be a diagonal matrix, then

$$\mathbf{U} = \begin{pmatrix} \tilde{a} + \partial_x \log \omega & \tilde{b} \omega^2 \\ \tilde{c} \omega^{-2} & \tilde{d} - \partial_x \log \omega \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \tilde{A} + \partial_t \log \omega & \tilde{B} \omega^2 \\ \tilde{C} \omega^{-2} & \tilde{D} - \partial_t \log \omega \end{pmatrix}. \quad (2.9)$$

Let us consider the two linear problems (2.1) in detail. Explicitly, we have:

$$\begin{cases} \partial_x \psi_1 = a \psi_1 + b \psi_2 \\ \partial_x \psi_2 = c \psi_1 + d \psi_2 \end{cases}, \quad \begin{cases} \partial_t \psi_1 = A \psi_1 + B \psi_2 \\ \partial_t \psi_2 = C \psi_1 + D \psi_2 \end{cases}$$

Applying ∂_x to the first equation of the first system, and using the second equation, we obtain

$$\partial_x^2 \psi_1 - (a + d) \partial_x \psi_1 + (ad - bc) \psi_1 - (a_x \psi_1 + b_x \psi_2) = 0. \quad (2.10)$$

Using the linear equations above, one can express ψ_2 through ψ_1 in two different ways:

$$\psi_2 = \frac{\partial_x \psi_1 - a \psi_1}{b} = \frac{\partial_t \psi_1 - A \psi_1}{B}. \quad (2.11)$$

The first possibility leads to a closed ordinary second-order differential equation for ψ_1 while the second one leads to a partial differential equation for ψ_1 as a function of x, t . As we shall see soon, both have the form of Schrödinger equations, stationary and non-stationary. This pair of scalar equations is equivalent to the original system (2.1) in the sense that their compatibility implies Painlevé equations for the dependent variable. One can also say that the second equation describes isomonodromic deformations of the first one. Let us consider them separately. From now on we will write simply ψ instead of ψ_1 .

2.2 Ordinary second-order differential equation

Using the first equality in (2.11), we get an ordinary second-order differential equation for $\psi := \psi_1$:

$$\partial_x^2 \psi - \left(a + d + \frac{b_x}{b} \right) \partial_x \psi + \left(ad - bc - a_x + \frac{b_x a}{b} \right) \psi = 0.$$

The coefficient functions here are expressed through entries of the matrix $\mathbf{U}(x, t)$. For traceless matrices with the condition (2.4) the equation acquires the form

$$\partial_x^2 \psi - \frac{b_x}{b} \partial_x \psi + \left(ad - bc - a_x + 2A - \frac{b_t}{b} + \frac{b_{xx}}{2b} \right) \psi = 0$$

(here for the transformation of the last term (2.5) has been used) or

$$\left(\frac{1}{2}\partial_x^2 - \frac{b_x}{2b}\partial_x + W(x)\right)\psi = 0, \quad (2.12)$$

where

$$W(x) = \frac{1}{2}(ad - bc - a_x + 2A) - \frac{1}{2b}\left(-\partial_t + \frac{1}{2}\partial_x^2\right)b. \quad (2.13)$$

The substitution $\psi = \sqrt{b}\check{\psi}$ kills the first derivative term in eq. (2.12) and brings it to the form of stationary Schrödinger equation

$$\left(\frac{1}{2}\partial_x^2 + \check{W}(x)\right)\check{\psi} = 0 \quad (2.14)$$

with the potential

$$\check{W} = W + \frac{1}{4}\partial_x^2 \log b - \frac{1}{8}(\partial_x \log b)^2. \quad (2.15)$$

This equation has formal solutions with the WKB-like asymptotes near poles of the potential:

$$\check{\psi}(x) \cong (-2\check{W})^{-\frac{1}{4}} e^{\pm \int^x \sqrt{-2\check{W}} dx'}. \quad (2.16)$$

An expansion of the right hand side near singularities of the potential allows one to extract solutions to the corresponding Painlevé equation.

2.3 Non-stationary Schrödinger equation

The second possibility in (2.11) is more interesting for us here. It leads to a partial differential equation for $\psi = \psi_1$ as a function of x, t :

$$\partial_x^2 \psi - (a + d)\partial_x \psi + (ad - bc)\psi - \left(a_x - \frac{b_x A}{B}\right)\psi - \frac{b_x}{B}\partial_t \psi = 0. \quad (2.17)$$

The coefficient functions here are expressed through entries of the both matrices $\mathbf{U}(x, t)$, $\mathbf{V}(x, t)$. For traceless matrices with the condition (2.4) the equation simplifies:

$$\partial_x^2 \psi + (ad - bc - a_x + 2A)\psi - 2\partial_t \psi = 0. \quad (2.18)$$

The role of the condition (2.4) is thus to make constant the coefficient in front of the time derivative (the specific value 2 of the constant is just a matter of normalization).

Equation (2.18) is central for what follows. Clearly, it has the form of a non-stationary Schrödinger equation in imaginary time:

$$\partial_t \psi = \left(\frac{1}{2}\partial_x^2 + U(x, t)\right)\psi \quad (2.19)$$

with the potential

$$U(x, t) = \frac{1}{2}(ad - bc - a_x) + A = \frac{1}{2}\det \mathbf{U} - \frac{a_x}{2} + A. \quad (2.20)$$

In the subsequent sections 3–8 we verify, by means of the case study, that for all Painlevé equations the dependent variable u enters this potential only through an irrelevant x -independent term while x -dependent terms contain the time variable in the explicit form only. Moreover, this potential turns out to be *the same as the classical mechanical potential for Painlevé equations written in the Calogero form*. (To be precise, we should point out that for higher members of the Painlevé family, $P_{IV} - P_{VI}$, the coefficients in front of different terms of the potential may be modified). This provides the quantum version of the Painlevé-Calogero correspondence.

Summing up, we have reduced the linear system (2.1) for the vector function $\Psi = (\psi_1, \psi_2)^t$ to two scalar equations for $\psi := \psi_1$:

$$\begin{cases} \left(\frac{1}{2} \partial_x^2 - \frac{1}{2} (\partial_x \log b) \partial_x + W(x, t) \right) \psi = 0 \\ \partial_t \psi = \left(\frac{1}{2} \partial_x^2 + U(x, t) \right) \psi. \end{cases} \quad (2.21)$$

The second equation describes isomonodromic deformations of the first one and their compatibility implies the Painlevé equation (in the Calogero form) for the function $u = u(t)$ defined as a (simple) zero of the function $b(x)$: $b(u) = 0$. The x -dependent part of the potential $U(x, t)$ does not contain the dependent variable u . Note that the potentials W and U are related by

$$W = U - \frac{1}{2} \partial_t \log b + \frac{1}{4} \partial_x^2 \log b + \frac{1}{4} (\partial_x \log b)^2,$$

so the potential $W(x, t)$ has an apparent singularity at $x = u(t)$.

One can see that equations (2.21) imply the scalar linear problems in the form suggested by R.Fuchs [2] and R.Garnier [15]. Indeed, passing to the function $\check{\psi} = \psi/\sqrt{b}$ and combining the two equations (2.21), one obtains the linear system

$$\begin{cases} \left(\frac{1}{2} \partial_x^2 + \check{W}(x, t) \right) \check{\psi} = 0 \\ \partial_t \check{\psi} = \left(\Lambda \partial_x - \frac{1}{2} (\partial_x \Lambda) \right) \check{\psi} \end{cases}, \quad \Lambda := \frac{1}{2} \partial_x \log b, \quad (2.22)$$

with \check{W} given by (2.15), which is exactly of the Fuchs-Garnier form. The integrability condition for this system is

$$\partial_t \check{W} = 2\check{W} \partial_x \Lambda + \Lambda \partial_x \check{W} + \frac{1}{4} \partial_x^3 \Lambda. \quad (2.23)$$

2.4 The linear problems and quantum Painlevé-Calogero correspondence

In this subsection we give a general view on what we are going to do in sections 3–8 for the particular Painlevé equations.

In the original form, the Painlevé equations can be written as

$$\partial_T^2 y = R(T, y, \partial_T y), \quad (2.24)$$

where R is a rational function of the independent variable T , the dependent variable y and its T -derivative. The Painlevé-Calogero correspondence means the existence of a change of variables from y, T to x, t of the form $y = y(x, t)$, $T = T(t)$ such that eq. (2.24) in the new variables acquires the form

$$\ddot{x} = -\partial_x V(x, t) \quad (2.25)$$

which is the Newton equation for motion of a point-like particle on the line in a time-dependent potential $V(x, t)$. In order to indicate the dependence on the parameters $\alpha, \beta, \gamma, \delta$ which may enter the Painlevé equations, we will write $V(x, t) = V^{(\alpha, \beta, \gamma, \delta)}(x, t)$. As it was already said in the Introduction, we call (2.25) *the Calogero form* of the Painlevé equation. Hereafter, the dot means the t -derivative. It should be noted that P_I and P_{II} equations are already of the Calogero form, so no change of the variables is necessary, for $P_{III} - P_V$ equations the transformation $y \rightarrow x$ bringing the equations to the Calogero form does not depend on t , and only for P_{VI} this transformation is actually t -dependent.

The linear problems of the necessary form described in section 2.1 have been known for lower members of the Painlevé family but not for higher ones (especially for P_V and P_{VI}). Therefore, we should start from a known version of the linear problems and then transform it to the desired form. A convenient starting point is the pair of compatible linear problems

$$\begin{cases} \partial_X \Psi = U(X, T) \Psi \\ \partial_T \Psi = V(X, T) \Psi \end{cases} \quad (2.26)$$

for a two-component vector function Ψ , where the matrices $U(X, T)$, $V(X, T)$ are rational functions of the spectral parameter X given in [18] for all the six Painlevé equations. The transformation from this pair of matrices to the pair of matrices $U(x, t)$, $V(x, t)$ with the desired properties will be done in two steps:

$$\{U(X, T), V(X, T)\} \xrightarrow{\mathcal{R}} \{\tilde{U}(x, t), \tilde{V}(x, t)\} \xrightarrow{\mathcal{G}} \{U(x, t), V(x, t)\}.$$

The transformation \mathcal{R} is a re-parametrization of the time and spectral parameter corresponding to the change of variables that prepares the Calogero form of the Painlevé equation from the original one. Here are some general relations for a change of variables from X, T to x, t of the form $X = X(x, t)$, $T = T(t)$. Clearly, such a change of variables implies the following relations for the partial derivatives:

$$\partial_x = \frac{\partial X}{\partial x} \partial_X, \quad \partial_t = \frac{\partial X}{\partial t} \partial_X + \frac{\partial T}{\partial t} \partial_T.$$

This means that the linear problems (8.6) are transformed as follows:

$$\begin{cases} \partial_x \Psi = \frac{\partial X}{\partial x} U \Psi \\ \partial_t \Psi = \left(\frac{\partial T}{\partial t} V + \frac{\partial X}{\partial t} U \right) \Psi. \end{cases} \quad (2.27)$$

Therefore, the U - V pair in the variables x, t is

$$\begin{aligned}\tilde{\mathbf{U}}(x, t) &= \frac{\partial X}{\partial x} \mathbf{U}(X(x, t), T(t)) \\ \tilde{\mathbf{V}}(x, t) &= \frac{\partial T}{\partial t} \mathbf{V}(X(x, t), T(t)) + \frac{\partial X}{\partial t} \mathbf{U}(X(x, t), T(t)),\end{aligned}\tag{2.28}$$

where the entries of the matrices \mathbf{U}, \mathbf{V} in the right hand side should be expressed through the new variables x, t . Note that we deliberately use the same letter x as in the equation of the classical motion (2.25) to stress the fact that it is this variable (x -coordinate of a particle on the line) which is going to be “quantized” in the “quantum” version of the Painlevé-Calogero correspondence in the sense that the momentum $p = \dot{x}$ is going to be replaced by the operator ∂_x . This analogy is justified by the final formulas.

The zero curvature condition for the pair of matrices $\tilde{\mathbf{U}}(x, t), \tilde{\mathbf{V}}(x, t)$ is equivalent to the Painlevé equation in the Calogero form

$$\ddot{u} = -\partial_u V(u, t)\tag{2.29}$$

for the function $u = u(t)$ defined as a (simple) zero of the right upper element $\tilde{b}(x, t) = \tilde{\mathbf{U}}_{12}(x, t)$ of the matrix $\tilde{\mathbf{U}}$: $\tilde{b}(u, t) = 0$. (To avoid a misunderstanding, we should stress that the time dependence of the function $u(t)$ is defined *not* by this equation but by the Painlevé equation.)

In general, the so obtained matrices $\tilde{\mathbf{U}}(x, t), \tilde{\mathbf{V}}(x, t)$ do not obey the condition (2.4). The transformation \mathcal{G} is a diagonal gauge transformation of the form (2.9) with a specially adjusted function $\omega(x, t)$ such that the condition (2.4) for the gauge-transformed matrices is satisfied. Here an important remark is in order. Given any two 2×2 matrix functions $\tilde{\mathbf{U}}(x, t), \tilde{\mathbf{V}}(x, t)$, one can always find a scalar function $\omega(x, t)$ such that the upper right entries of the gauge-transformed matrices, $b = \tilde{b}\omega^2 = \tilde{\mathbf{U}}_{12}\omega^2$, $B = \tilde{B}\omega^2 = \tilde{\mathbf{V}}_{12}\omega^2$, are related by the equation $b_x = 2B$. Indeed, such a function ω can be found as a solution to the differential equation

$$\partial_x \log \omega = \frac{\tilde{B}}{\tilde{b}} - \frac{1}{2} \partial_x \log \tilde{b}.$$

A non-trivial additional constraint on the function ω is that it should factorize into a product of two functions such that one of them depends on x, t but *does not contain the dependent variable u* and another one depends on t only (through both dependent and independent variables). In fact this is a necessary condition for the perfect classical-quantum correspondence. Otherwise the spectral parameter and the dependent variable have no chance to separate in the potential of the non-stationary Schrödinger equation. In fact the example of P_{VI} shows that the two transformations, \mathcal{R} and \mathcal{G} , should be found simultaneously from the condition that the function ω be of the special form in which the dependent variable separates from the spectral parameter.

The resulting pair of matrices $\mathbf{U}(x, t), \mathbf{V}(x, t)$ is the one that was discussed in section 2.1. The zero curvature condition for these matrices is equivalent to the Painlevé equation (2.29) for the function $u = u(t)$ which can be equivalently defined as a (simple) zero of the right upper element $b(x, t) = \mathbf{U}_{12}(x, t)$ of the matrix \mathbf{U} : $b(u, t) = 0$. One can also check that the value of the diagonal element, $\mathbf{U}_{11}(x, t)$, at $x = u$ is a canonically conjugate

variable to u , in accordance with the general constructions of [33, 34]. (A more detailed discussion of this point will be given elsewhere.)

Further, we are going to reduce the system of linear problems (2.1) to the pair of scalar Schrödinger-like equations (2.21) according to the procedure outlined in sections 2.2 and 2.3. The result merits attention and further understanding from “first principles”. The explicit calculations in each case show that for any Painlevé equation (with possible parameters $\alpha, \beta, \gamma, \delta$) the following holds true:

- The variables x, u separate in the non-stationary Schrodinger equation meaning that

$$U(x, t) = V^{(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})}(x, t) - H^{(\alpha, \beta, \gamma, \delta)}(\dot{u}, u), \quad (2.30)$$

where the potential $V^{(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})}(x, t)$ is of the same form as the one for the classical equation (2.25) (or (2.29)) with possibly modified parameters and the x -independent term, $H^{(\alpha, \beta, \gamma, \delta)}(\dot{u}, u)$, is the classical Hamiltonian

$$H(\dot{u}, u) = H^{(\alpha, \beta, \gamma, \delta)}(\dot{u}, u) = \frac{1}{2} \dot{u}^2 + V^{(\alpha, \beta, \gamma, \delta)}(u, t)$$

for the Painlevé equation in the Calogero form;

- For P_I – P_{III} the parameters in the quantum Hamiltonian are the same as in the classical one while for P_{IV} – P_{VI} some or all parameters should be shifted: $(\tilde{\alpha}, \tilde{\beta}) = (\alpha, \beta + \frac{1}{2})$ for P_{IV} , $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = (\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma, \delta)$ for P_V and $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = (\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma - \frac{1}{8}, \delta + \frac{1}{8})$ for P_{VI} .

This means that the function

$$\Psi(x, t) = e^{\int^t H(\dot{u}, u) dt'} \psi(x, t) \quad (2.31)$$

is a common solution to the linear differential equations

$$\begin{cases} \left(\frac{1}{2} \partial_x^2 - \frac{1}{2} (\partial_x \log b) \partial_x + W(x, t) \right) \Psi = 0 \\ \partial_t \Psi = \left(\frac{1}{2} \partial_x^2 + V(x, t) \right) \Psi. \end{cases} \quad (2.32)$$

The second one is the non-stationary Schrödinger equation $\partial_t \Psi = H(\partial_x, x) \Psi$ whose Hamiltonian is the natural quantization of the classical Hamiltonian of the Painlevé equation, possibly with modified parameters (such a modification, if any, can be regarded as a “quantum correction”). This is what we call the quantum Painlevé-Calogero correspondence or the classical-quantum correspondence for the Painlevé equations.

3 Painlevé I

3.1 The equation

The P_I equation

$$4\ddot{x} = 6x^2 + t \quad (3.1)$$

is already of the Calogero form from the very beginning, so no change of variables is necessary in this case. It can be written in the standard Hamiltonian form as

$$\dot{x} = \frac{\partial H_I}{\partial p}, \quad \dot{p} = -\frac{\partial H_I}{\partial x}$$

with the classical time-dependent Hamiltonian

$$H_I = H_I(p, x) = \frac{p^2}{2} - \frac{x^3}{2} - \frac{tx}{4}. \quad (3.2)$$

One may introduce the potential

$$V_I(x) = -\frac{x^3}{2} - \frac{tx}{4}, \quad (3.3)$$

then the P_I equation takes the form $\ddot{x} = -\partial_x V_I(x)$ which is the Newton equation for a point-like particle on the line in the time-dependent potential. Note that the partial and full time derivatives of the Hamiltonian coincide:

$$\frac{\partial H_I}{\partial t} = \frac{dH_I}{dt} = -\frac{x(t)}{4} \quad (3.4)$$

(the first equality is of course the general property of Hamiltonians for non-conservative systems while the second one is specific for the P_I equation).

3.2 Linearization and classical-quantum correspondence for P_I

In the case of P_I the general construction outlined in section 2 is especially simple and transparent because it does not need neither the change of variables nor the gauge transformation. The P_I equation $4\ddot{u} = 6u^2 + t$ is known to be the compatibility condition for the linear problems (2.1) with the matrices

$$\mathbf{U}(x, t) = \begin{pmatrix} \dot{u} & x - u \\ x^2 + xu + u^2 + \frac{1}{2}t & -\dot{u} \end{pmatrix}, \quad \mathbf{V}(x, t) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2}x + u & 0 \end{pmatrix} \quad (3.5)$$

which are already of the form implied in section 2.1. Note that u is the simple zero of the right upper element of the matrix $\mathbf{U}(x, t)$: $b(u) = 0$.

Another meaning of the P_I equation (which we will not discuss here) is the condition that the monodromy data of the first linear problem in (2.1) be independent of the parameter t .

The spectral parameter is denoted by x . We deliberately use the same letter x as in the equation of the classical motion (3.1) to stress the fact that it is this variable (x -coordinate of a particle on the line) which is going to be “quantized” in the “quantum” version of the Painlevé-Calogero correspondence in the sense that the momentum $p = \dot{x}$ is going to be replaced by the operator ∂_x . The notation with the same idea in mind will be used below for other Painlevé equations.

It remains to apply the general formulas of section 2. Consider equation (2.19). In the case of P_I

$$ad - bc = -x^3 - \frac{tx}{2} - \dot{u}^2 + u^3 + \frac{tu}{2}, \quad a_x = A = 0.$$

so the calculation of the potential $U(x, t)$ is very simple. As a result, we obtain the non-stationary Schrödinger equation (in imaginary time)

$$\partial_t \psi = \left(\frac{1}{2} \partial_x^2 - \frac{x^3}{2} - \frac{tx}{4} - H_I(\dot{u}, u) \right) \psi, \quad (3.6)$$

where $H_I(\dot{u}, u)$ is given by (3.2). We can write it in the form

$$\partial_t \psi = \left(H_I(\partial_x, x) - H_I(\dot{u}, u) \right) \psi \quad (3.7)$$

where

$$H_I(\partial_x, x) = \frac{1}{2} \partial_x^2 - \frac{x^3}{2} - \frac{tx}{4}$$

is the *quantum* Hamiltonian operator obtained as a literal quantization of the classical Hamiltonian (3.2). The function

$$\Psi(x, t) = e^{\int^t H_I(\dot{u}, u) dt'} \psi(x, t) \quad (3.8)$$

thus obeys the non-stationary Schrödinger equation

$$\partial_t \Psi = H_I(\partial_x, x) \Psi = \left(\frac{1}{2} \partial_x^2 + V_I(x, t) \right) \Psi \quad (3.9)$$

without a free t -dependent term.

To conclude, we have two equivalent representations of the P_I equation. One is a classical motion in the time-dependent cubic potential with Hamiltonian (3.2). The coordinate of the particle as a function of time obeys the P_I equation. Another representation is a time-dependent quantum mechanical particle in the same time-dependent potential. The non-stationary Schrödinger equation for this quantum system in the coordinate representation simultaneously serves as the linear problem for time evolution associated with the Painlevé equation¹.

4 Painlevé II

4.1 The equation

The P_{II} equation

$$\ddot{x} = 2x^3 + tx - \alpha, \quad (4.1)$$

¹As we learned from B.Suleimanov after completion of this work, this fact was pointed out in [30, 31], see also [32].

where α is an arbitrary parameter, is already of the Calogero form from the very beginning, so no change of variables is necessary in this case. It can be written in the standard Hamiltonian form as

$$\dot{x} = \frac{\partial H_{\text{II}}}{\partial p}, \quad \dot{p} = -\frac{\partial H_{\text{II}}}{\partial x}$$

with the classical time-dependent Hamiltonian

$$\begin{aligned} H_{\text{II}} = H_{\text{II}}(p, x) &= \frac{p^2}{2} - \frac{1}{2} \left(x^2 + \frac{t}{2} \right)^2 + \alpha x \\ &= \frac{p^2}{2} - \frac{x^4}{2} - \frac{tx^2}{2} - \frac{t^2}{8} + \alpha x. \end{aligned} \tag{4.2}$$

One may introduce the potential

$$V_{\text{II}}(x) = -\frac{1}{2} \left(x^2 + \frac{t}{2} \right)^2 + \alpha x, \tag{4.3}$$

then the P_{II} equation takes the Newton form $\ddot{x} = -\partial_x V_{\text{II}}(x)$. Note that the partial and full time derivatives of the Hamiltonian coincide:

$$\frac{\partial H_{\text{II}}}{\partial t} = \frac{dH_{\text{II}}}{dt} = -\frac{x^2(t)}{2} - \frac{t}{4} \tag{4.4}$$

(again, the first equality is a general property of Hamiltonians for non-conservative systems while the second one is specific for the P_{II} equation).

4.2 Linearization and classical-quantum correspondence for P_{II}

The linear problems and their compatibility condition for the P_{II} equation

$$\ddot{u} = 2u^3 + tu - \alpha$$

are given by (2.1), (2.2) with the matrices

$$\mathbf{U} = \begin{pmatrix} x^2 + \dot{u} - u^2 & x - u \\ (x + u)(2u^2 - 2\dot{u} + t) - 2\alpha - 1 & -x^2 - \dot{u} + u^2 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \frac{x+u}{2} & \frac{1}{2} \\ u^2 - \dot{u} + \frac{t}{2} & -\frac{x+u}{2} \end{pmatrix}. \tag{4.5}$$

They are of the form implied in section 2.1. Note that u is the simple zero of the right upper element of the matrix $\mathbf{U}(x, t)$: $b(u) = 0$.

The spectral parameter is again deliberately denoted by the same letter x as in the equation of classical motion (4.1) to stress the fact that it is this variable (x -coordinate of a particle on the line) which is going to be “quantized” in the “quantum” version of the Painlevé-Calogero correspondence in the sense that the momentum $p = \dot{x}$ is going to be replaced by the operator ∂_x .

Another meaning of the P_{II} equation (which we will not discuss here) is the condition that the monodromy data of the first linear problem in (2.1) be independent of the parameter t .

It remains to calculate the potential $U(x, t)$ of the non-stationary Schrödinger equation. In the case of P_{II}

$$ad - bc = -x^4 - tx^2 + (2\alpha + 1)x - \dot{u}^2 + u^4 + tu^2 - (2\alpha + 1)u,$$

$$a_x - b_x A/B = x - u.$$

As a result, we obtain the non-stationary Schrödinger equation (in imaginary time)

$$\partial_t \psi = \left(\frac{1}{2} \partial_x^2 - \frac{x^4}{2} - \frac{tx^2}{2} + \alpha x - \frac{t^2}{8} - H_{II}(\dot{u}, u) \right) \psi \quad (4.6)$$

or

$$\partial_t \psi = \left(H_{II}(\partial_x, x) - H_{II}(\dot{u}, u) \right) \psi, \quad (4.7)$$

where

$$H_{II}(\partial_x, x) = \frac{1}{2} \partial_x^2 - \frac{x^4}{2} - \frac{tx^2}{2} + \alpha x - \frac{t^2}{8}$$

is the *quantum* Hamiltonian operator obtained as a literal quantization of the classical Hamiltonian (4.2). The function

$$\Psi(x, t) = e^{\int^t H_{II}(\dot{u}, u) dt'} \psi(x, t) \quad (4.8)$$

thus obeys the non-stationary Schrödinger equation

$$\partial_t \Psi = H_{II}(\partial_x, x) \Psi = \left(\frac{1}{2} \partial_x^2 + V_{II}(x, t) \right) \Psi \quad (4.9)$$

without a free t -dependent term.

To conclude, we have two equivalent representations of the P_{II} equation. One is a classical motion in the time-dependent polynomial potential with Hamiltonian (4.2). The coordinate of the particle as a function of time obeys the P_{II} equation. Another representation is a quantum mechanical particle in the same time-dependent potential. The non-stationary Schrödinger equation for this quantum system in the coordinate representation simultaneously serves as the linear problem for time evolution associated with the Painlevé equation.

5 Painlevé IV

5.1 The equation

The standard form of the Painlevé IV (P_{IV}) equation is

$$\partial_t^2 y = \frac{(\partial_t y)^2}{2y} + \frac{3}{2} y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \quad (5.1)$$

where α, β are arbitrary parameters. This is the first example where a change of variable is necessary. The time variable t remains the same but the dependent variable should be changed as $y = x^2$. This brings the equation to the Newton form

$$\ddot{x} = \frac{3}{4} x^5 + 2tx^3 + (t^2 - \alpha)x + \frac{\beta}{2x^3} \quad (5.2)$$

which admits a Hamiltonian structure similar to the previous examples:

$$\dot{x} = \frac{\partial H_{\text{IV}}^{(\alpha,\beta)}}{\partial p}, \quad \dot{p} = -\frac{\partial H_{\text{IV}}^{(\alpha,\beta)}}{\partial x}$$

with the classical time-dependent Hamiltonian

$$H_{\text{IV}}^{(\alpha,\beta)} = H_{\text{IV}}^{(\alpha,\beta)}(p, x) = \frac{p^2}{2} - \frac{x^6}{8} - \frac{tx^4}{2} - \frac{1}{2}(t^2 - \alpha)x^2 + \frac{\beta}{4x^2}. \quad (5.3)$$

One may introduce the potential

$$V_{\text{IV}}(x) = V_{\text{IV}}^{(\alpha,\beta)}(x) - \frac{x^6}{8} - \frac{tx^4}{2} - \frac{1}{2}(t^2 - \alpha)x^2 + \frac{\beta}{4x^2}, \quad (5.4)$$

then the P_{IV} equation in the Calogero form (5.2) reads $\ddot{x} = -\partial_x V_{\text{IV}}(x)$. Note that the partial and full time derivatives of the Hamiltonian coincide:

$$\frac{\partial H_{\text{IV}}^{(\alpha,\beta)}}{\partial t} = \frac{dH_{\text{IV}}^{(\alpha,\beta)}}{dt} = -\frac{x^4(t)}{2} - tx^2(t) \quad (5.5)$$

(again, the first equality is a general property of Hamiltonians for non-conservative systems while the second one is specific for the P_{IV} equation).

5.2 Linearization and classical-quantum correspondence for P_{IV}

The system of linear problems associated with the P_{IV} equation for the u -variable in the Calogero form,

$$\ddot{u} = \frac{3}{4}u^5 + 2tu^3 + (t^2 - \alpha)u + \frac{\beta}{2u^3}, \quad (5.6)$$

is a modified version of the one given in [36]. Their compatibility condition is of the same form (2.2) with

$$\mathbf{U} = \begin{pmatrix} \frac{x^3}{2} + tx + \frac{Q + \frac{1}{2}}{x} & x^2 - u^2 \\ \frac{Q^2 + \frac{\beta}{2}}{u^2 x^2} - Q - \alpha - 1 & -\frac{x^3}{2} - tx - \frac{Q + \frac{1}{2}}{x} \end{pmatrix} \quad (5.7)$$

$$\mathbf{V} = \begin{pmatrix} \frac{x^2 + u^2}{2} + t & x \\ -\frac{Q + \alpha + 1}{x} & -\frac{x^2 + u^2}{2} - t \end{pmatrix}, \quad (5.8)$$

where

$$Q = u\dot{u} - \frac{u^4}{2} - tu^2.$$

Note that these matrices enjoy the property $b_x = 2B$ and, therefore, the non-stationary Schrödinger equation of the form (2.19) is valid. (Equivalently, we could start from a rational U - V pair given in [18]) and transform it to the desired form according to the

strategy outlined in section 2.4 but in this case the transformations are simple enough and do not require any special consideration.) Note also that u is one of the two simple zeros of the right upper element of the matrix $\mathbf{U}(x, t)$: $b(u) = 0$. The second zero at the point $x = -u$ leads to the same results because the equation (5.6) is invariant under the transformation $u \rightarrow -u$.

Again, we deliberately denote the spectral parameter by the same letter x as in the equation of classical motion (5.2) to stress the fact that it is this variable (x -coordinate of a particle on the line) which is going to be “quantized” in the “quantum” version of the Painlevé-Calogero correspondence in the sense that the momentum $p = \dot{x}$ is going to be replaced by the operator ∂_x .

Let us calculate the potential $U(x, t)$ in equation (2.19). It consists of two parts: one of them is one half of the determinant of the matrix \mathbf{U} and another one is $-\frac{1}{2}a_x + A$. For clarity, we present the results for these two parts separately and then take the sum. The calculation of the determinant yields:

$$\begin{aligned} ad - bc &= \frac{x^6}{4} - tx^4 - \left(t^2 - \alpha - \frac{1}{2}\right)x^2 + \frac{\beta - \frac{1}{2}}{2x^2} \\ &\quad - \left(\dot{u}^2 - \frac{u^6}{4} - tu^4 - (t^2 - \alpha - 1)u^2 + \frac{\beta}{2u^2}\right) - t - \frac{Q}{x^2}. \end{aligned}$$

We see that the variables x and u do not completely separate in this expression because of the last term (recall that Q is not a constant but a dynamical variable). Fortunately, this term cancels out after adding the second part of the potential:

$$-a_x + 2A = -\frac{x^2}{2} + \frac{1}{2x^2} + u^2 + t + \frac{Q}{x^2}.$$

Combining the two parts together, we get:

$$\begin{aligned} \frac{1}{2}(ad - bc - a_x + 2A) &= \frac{x^6}{8} - \frac{tx^4}{2} - \frac{1}{2}(t^2 - \alpha)x^2 + \frac{\beta + \frac{1}{2}}{4x^2} \\ &\quad - \left(\frac{\dot{u}^2}{2} - \frac{u^6}{8} - \frac{tu^4}{2} - \frac{1}{2}(t^2 - \alpha)u^2 + \frac{\beta}{4u^2}\right). \end{aligned} \tag{5.9}$$

Therefore, equation (2.19) reads

$$\partial_t \psi = \left(H_{\text{IV}}^{(\alpha, \beta + \frac{1}{2})}(\partial_x, x) - H_{\text{IV}}^{(\alpha, \beta)}(\dot{u}, u)\right)\psi, \tag{5.10}$$

where

$$H_{\text{IV}}^{(\alpha, \beta + \frac{1}{2})}(\partial_x, x) = \frac{1}{2}\partial_x^2 - \frac{x^6}{8} - \frac{tx^4}{2} - \frac{1}{2}(t^2 - \alpha)x^2 + \frac{\beta + \frac{1}{2}}{4x^2}.$$

The function

$$\Psi(x, t) = e^{\int^t H_{\text{IV}}^{(\alpha, \beta)}(\dot{u}, u) dt'} \psi(x, t) \tag{5.11}$$

thus obeys the non-stationary Schrödinger equation

$$\partial_t \Psi = H_{\text{IV}}^{(\alpha, \beta + \frac{1}{2})}(\partial_x, x)\Psi = \left(\frac{1}{2}\partial_x^2 + V_{\text{IV}}^{(\alpha, \beta + \frac{1}{2})}(x, t)\right)\Psi \tag{5.12}$$

without a free t -dependent term. Note the shift $\beta \rightarrow \beta + \frac{1}{2}$ which can be thought of as a “quantum correction”.

To conclude, we have two equivalent representations of the P_{IV} equation. One is a classical motion in the time-dependent potential with Hamiltonian (5.3). The coordinate of the particle as a function of time obeys the P_{IV} equation. Another representation is a quantum mechanical particle in the time-dependent potential of the same form, with the modified coefficient in front of $1/x^2$. The non-stationary Schrödinger equation for this quantum system in the coordinate representation simultaneously serves as the linear problem for time evolution associated with the Painlevé equation.

6 Painlevé III

6.1 The equation

The standard form of the P_{III} equation for a function $y(T)$ is

$$\partial_T^2 y = \frac{(\partial_T y)^2}{y} - \frac{\partial_T y}{T} + \frac{1}{T}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \quad (6.1)$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters. The change of the variables $T = e^t, y = e^{2x}$ brings this equation to the Newton form

$$2\ddot{x} = e^t(\alpha e^{2x} + \beta e^{-2x}) + e^{2t}(\gamma e^{4x} + \delta e^{-4x}). \quad (6.2)$$

Note that only two parameters of the four are really independent because the other two can be eliminated by the shifts $x \rightarrow x_0, t \rightarrow t_0$ with constant x_0, t_0 . However, we will keep 3 parameters in order to be able to consider some particular cases which are not reachable otherwise. So, equation (6.2) acquires the form

$$\ddot{x} = 2\nu^2 e^t \sinh(2x - 2\rho) + 4\mu^2 e^{2t} \sinh(4x), \quad (6.3)$$

where ν, μ, ρ are the parameters. In principle, one of them, say ν can be put equal to one by the shift $t \rightarrow t - 2 \log \nu$ but this works only if $\nu \neq 0$.

Equation (6.3) admits a Hamiltonian structure similar to the previous examples:

$$\dot{x} = \frac{\partial H_{III}}{\partial p}, \quad \dot{p} = -\frac{\partial H_{III}}{\partial x}$$

with the classical time-dependent Hamiltonian

$$H_{III} = H_{III}(p, x) = \frac{p^2}{2} - \nu^2 e^t \cosh(2x - 2\rho) - \mu^2 e^{2t} \cosh(4x). \quad (6.4)$$

One may introduce the potential

$$V_{III}(x) = -\nu^2 e^t \cosh(2x - 2\rho) - \mu^2 e^{2t} \cosh(4x), \quad (6.5)$$

then the P_{III} equation reads $\ddot{x} = -\partial_x V_{III}(x)$.

The case $\mu = 0$ is special. In this case, one can put $\varrho = 0$ without loss of generality, so the P_{III} equation acquires the form

$$\ddot{x} = 2\nu^2 e^t \sinh(2x) \quad (6.6)$$

with just one parameter ν (which in fact can be eliminated by a shift of time) and with the classical Hamiltonian

$$H_{\text{III}} = \frac{p^2}{2} - \nu^2 e^t \cosh(2x). \quad (6.7)$$

This equation will be referred to as truncated P_{III} equation.

6.2 The U - V pairs for P_{III}

6.2.1 The case of the truncated P_{III} equation

The truncated P_{III} equation

$$\ddot{u} = 2\nu^2 e^t \sinh(2u) \quad (6.8)$$

is the compatibility condition for linear problems with matrices \mathbf{U}, \mathbf{V} of rather simple form. Indeed, it is easy to check that the zero curvature condition (2.2) with

$$\mathbf{U}(x, t) = \begin{pmatrix} \dot{u} & 2\nu e^{t/2} \sinh(x - u) \\ 2\nu e^{t/2} \sinh(x + u) & -\dot{u} \end{pmatrix}, \quad (6.9)$$

$$\mathbf{V}(x, t) = \begin{pmatrix} 0 & \nu e^{t/2} \cosh(x - u) \\ \nu e^{t/2} \cosh(x + u) & 0 \end{pmatrix} \quad (6.10)$$

yields equation (6.8)². Moreover, these matrices obviously satisfy the condition $b_x = 2B$ and u is the first order zero of the element $b(x)$ (of course there are infinitely many zeros in the complex x -plane at the points of the lattice $u + \pi i\mathbb{Z}$ but all of them obey the same equation (6.8)).

6.2.2 The general case

In the general case the U - V pair for the P_{III} equation is more complicated. We take the linear problems for P_{III} given in [18] as a starting point, passing to the exponential parametrization from the very beginning and then transform them to the ones appropriate for our purpose.

So, we start with the linear problems

$$\partial_x \tilde{\Psi} = \begin{pmatrix} e^{2x+t} - g_{11}e^{-2x+t} + \theta + \frac{1}{2} & 2\nu e^{x-\frac{t}{2}} - g_{12}e^{-x+\frac{t}{2}} \\ 2\nu e^{-x+\frac{t}{2}} - g_{21}e^{-3x+\frac{3t}{2}} & -e^{2x+t} + g_{11}e^{-2x+t} - \theta - \frac{1}{2} \end{pmatrix} \tilde{\Psi} \quad (6.11)$$

²The Lax pairs for P_{III} and P_{V} were obtained by G.Aminov and S.Arthamonov via trigonometric scaling limits from the one found in [40]. However, the condition $b_x = 2B$ for the Lax pairs obtained in this way holds in the case of the truncated P_{III} equation only and does not hold in general.

$$\partial_t \tilde{\Psi} = \frac{1}{2} \begin{pmatrix} e^{2x+t} + g_{11}e^{-2x+t} - \frac{1}{2} & 2ve^{x-\frac{t}{2}} + g_{12}e^{-x+\frac{t}{2}} \\ 2we^{-x+\frac{t}{2}} + g_{21}e^{-3x+\frac{3t}{2}} & -e^{2x+t} - g_{11}e^{-2x+t} + \frac{1}{2} \end{pmatrix} \tilde{\Psi} \quad (6.12)$$

where $v, w, g_{11}, g_{12}, g_{21}$ are yet unknown functions of t and θ is a parameter. The functions g_{ik} are naturally thought of as entries of a traceless matrix

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & -g_{11} \end{pmatrix}. \quad (6.13)$$

Note that the x -derivative of the right upper element of the $\tilde{\mathbf{U}}$ -matrix in (6.11) is just equal to twice the right upper element of the $\tilde{\mathbf{V}}$ -matrix in (6.12).

As before, we deliberately denote the spectral parameter by the same letter x as in the equation of the classical motion (6.2) to stress the fact that it is this variable (x -coordinate of a particle on the line) which is going to be “quantized” in the “quantum” version of the Painlevé-Calogero correspondence in the sense that the momentum $p = \dot{x}$ is going to be replaced by the operator ∂_x .

The compatibility of the linear problems (6.11), (6.12) implies the following system of differential equations:

$$\begin{cases} \dot{g}_{11} = 2(vg_{21} - wg_{12}) \\ \dot{g}_{12} = \theta g_{12} - 4vg_{11} \\ \dot{g}_{21} = -\theta g_{21} + 4wg_{11} \\ \dot{v} = -\theta v - g_{12}e^{2t} \\ \dot{w} = \theta w + g_{21}e^{2t}. \end{cases} \quad (6.14)$$

Combining these equations, one easily finds two integrals:

$$\chi := g_{11}^2 + g_{12}g_{21} \quad (6.15)$$

$$\lambda := vg_{21} + wg_{12} + \theta g_{11}, \quad (6.16)$$

where χ and λ are integration constants. Note that the first integral is just determinant of the matrix G (6.13) with opposite sign.

Using (6.15), (6.16), one can exclude w and g_{21} ,

$$w = \frac{\lambda - \theta g_{11} - vg_{21}}{g_{12}}, \quad g_{21} = \frac{\chi - g_{11}^2}{g_{12}},$$

and reduce the system (6.14) to a simpler one:

$$\begin{cases} \dot{g}_{11} = 4vg_{12}^{-1}(\chi - g_{11}^2) + 2\theta g_{11} - 2\lambda \\ \dot{g}_{12} = \theta g_{12} - 4vg_{11} \\ \dot{v} = -\theta v - g_{12}e^{2t}. \end{cases} \quad (6.17)$$

Further, these equations imply the following system for the functions $f = v/g_{12}$, $g = g_{11}$:

$$\begin{cases} \dot{f} = 4gf^2 - 2\theta f - e^{2t} \\ \dot{g} = -4fg^2 + 2\theta g + 4\chi f - 2\lambda. \end{cases} \quad (6.18)$$

Now, substituting

$$g = \frac{\dot{f} + 2\theta f + e^{2t}}{4f^2}$$

from the first equation into the second one, we obtain a closed equation for f ,

$$\ddot{f} = \frac{\dot{f}^2}{f} + 16\chi f^3 - 8\lambda f^2 - 2(\theta+1)e^{2t} - \frac{e^{4t}}{f} \quad (6.19)$$

which is equivalent to the P_{III} equation (6.1) and can be brought to the original form by the change of variable $T = e^t$. The change of the dependent variable $f = e^{-2u+t}$ yields the equation

$$\ddot{u} = e^t \left((\theta+1)e^{2u} + 4\lambda e^{-2u} \right) + \frac{1}{2} e^{2t} \left(e^{4u} - 16\chi e^{-4u} \right) \quad (6.20)$$

which has the form (6.3) with $\mu = 1/2$ under the identification of parameters

$$\theta + 1 = \nu^2 e^{-2\varrho}, \quad 4\lambda = -\nu^2 e^{2\varrho}, \quad \chi = \frac{1}{16}.$$

6.3 Classical-quantum correspondence for P_{III}

6.3.1 The case of truncated P_{III} equation

Let us start with the simplest case $\mu = 0$ and use the U - V pair (6.9), (6.10). A simple calculation shows that in this case the linear equation for ψ (2.17) becomes the “non-stationary Mathieu equation”

$$\partial_t \psi = \left(H_{\text{III}}(\partial_x, x) - H_{\text{III}}(\dot{u}, u) \right) \psi, \quad (6.21)$$

where

$$H_{\text{III}}(\partial_x, x) = \frac{1}{2} \partial_x^2 - \nu^2 e^t \cosh(2x),$$

i.e., we again observe a perfect classical-quantum correspondence. Note that in this case $a_x = A = 0$, so the potential is given solely by determinant of the matrix \mathbf{U} :

$$\begin{aligned} \frac{1}{2}(ad - bc) &= -\frac{\dot{u}^2}{2} - 2\nu^2 e^t \sinh(x+u) \sinh(x-u) \\ &= -\nu^2 e^t \cosh(2x) - \left(\frac{\dot{u}^2}{2} - \nu^2 e^t \cosh(2u) \right). \end{aligned}$$

We remark that the non-stationary Mathieu equation in connection with the P_{III} equation was mentioned in [42].

6.3.2 The general case

In order to achieve a precise classical-quantum correspondence in the general case of the P_{III} equation with arbitrary parameters, one should modify the system of linear problems given above by a diagonal x -independent (but t -dependent) gauge transformation of the form (2.9) with

$$\omega = g_{12}^{-\frac{1}{2}} (2f e^{-t})^{-\frac{1}{4}}, \quad (6.22)$$

where $f = v/g_{12}$ as before.

Another small modification which is necessary to achieve perfect classical-quantum correspondence is the shift of the spectral parameter $x \rightarrow x - \frac{1}{2} \log 2$. Then the linear problems (6.11), (6.12) acquire the form (2.1) with

$$\mathbf{U} = \begin{pmatrix} \frac{1}{2} e^{2x+t} - 2g_{11} e^{-2x+t} + \theta + \frac{1}{2} & f^{\frac{1}{2}} e^x - f^{-\frac{1}{2}} e^{-x+t} \\ 4(vg_{12})^{\frac{1}{2}} (w e^{-x} - g_{21} e^{-3x+t}) & -\frac{1}{2} e^{2x+t} + 2g_{11} e^{-2x+t} - \theta - \frac{1}{2} \end{pmatrix} \quad (6.23)$$

$$\mathbf{V} = \begin{pmatrix} \frac{1}{4} e^{2x+t} + g_{11} e^{-2x+t} + h & \frac{1}{2} (f^{\frac{1}{2}} e^x + f^{-\frac{1}{2}} e^{-x+t}) \\ 2(vg_{12})^{\frac{1}{2}} (w e^{-x} + g_{21} e^{-3x+t}) & -\frac{1}{4} e^{2x+t} - g_{11} e^{-2x+t} - h \end{pmatrix}, \quad (6.24)$$

where

$$h := \partial_t \log \left(f^{-\frac{1}{4}} g_{12}^{-\frac{1}{2}} \right) = \frac{\dot{f}}{4f} + \frac{e^{2t}}{2f} + \frac{\theta}{2}. \quad (6.25)$$

Recall also that

$$f = e^{-2u+t},$$

so the right upper element of the matrix \mathbf{U} is $b(x) = 2e^{t/2} \sinh(x - u)$ and u is its first order zero.

Now, the calculation of the potential $U(x, t)$ in the non-stationary Schrödinger equation (2.19) yields

$$\begin{aligned} U(x, t) &= -\frac{e^{2t}}{8} (e^{4x} + 16\chi e^{-4x}) - \frac{e^t}{2} ((\theta+1)e^{2x} - 4\lambda e^{-2x}) \\ &\quad - \frac{\dot{u}^2}{2} + \frac{e^{2t}}{8} (e^{4u} + 16\chi e^{-4u}) + \frac{e^t}{2} ((\theta+1)e^{2u} - 4\lambda e^{-2u}), \end{aligned} \quad (6.26)$$

so the Schrödinger equation acquires the desired form

$$\partial_t \psi = \left(H_{\text{III}}(\partial_x, x) - H_{\text{III}}(\dot{u}, u) \right) \psi, \quad (6.27)$$

where

$$H_{\text{III}}(\partial_x, x) = \frac{1}{2} \partial_x^2 - \frac{e^{2t}}{8} (e^{4x} + 16\chi e^{-4x}) - \frac{e^t}{2} ((\theta+1)e^{2x} - 4\lambda e^{-2x}).$$

The function

$$\Psi(x, t) = e^{\int^t H_{\text{III}}(\dot{u}, u) dt'} \psi(x, t) \quad (6.28)$$

thus obeys the non-stationary Schrödinger equation

$$\partial_t \Psi = H_{\text{III}}(\partial_x, x) \Psi = \left(\frac{1}{2} \partial_x^2 + V_{\text{III}}(x, t) \right) \Psi \quad (6.29)$$

with the same classical potential (6.5).

To conclude, we have two equivalent representations of the P_{III} equation. One is a classical motion in the time-dependent potential with Hamiltonian (6.4). The coordinate of the particle as a function of time obeys the P_{III} equation. Another representation is a quantum mechanical particle in the same time-dependent potential described by a non-stationary Schrödinger equation. The latter simultaneously serves as the linear problem for time evolution associated with the Painlevé equation.

7 Painlevé V

7.1 The equation

The standard form of the P_V equation is

$$\partial_T^2 y = \left(\frac{1}{2y} + \frac{1}{y-1} \right) (\partial_T y)^2 - \frac{\partial_T y}{T} + \frac{y(y-1)^2}{T^2} \left(\alpha + \frac{\beta}{y^2} + \frac{\gamma T}{(y-1)^2} + \frac{\delta T^2(y+1)}{(y-1)^3} \right), \quad (7.1)$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters. A re-scaling of the dependent variable allows one to fix one of these parameters, so there are three essentially independent parameters. The change of the time variable $T = e^{2t}$ allows one to eliminate the term $\partial_T y / T$ in the right hand side, so that the equation becomes

$$\ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + 4(y-1)^2 \left(\alpha y + \frac{\beta}{y} \right) + 4\gamma e^{2t} y + \frac{4\delta e^{4t} y(y+1)}{y-1}. \quad (7.2)$$

Further, the change of the dependent variable

$$y = \coth^2 x \quad (7.3)$$

brings the equation to the form

$$\ddot{x} = -\frac{2\alpha \cosh x}{\sinh^3 x} - \frac{2\beta \sinh x}{\cosh^3 x} - \gamma e^{2t} \sinh(2x) - \frac{1}{2} \delta e^{4t} \sinh(4x) \quad (7.4)$$

which can be written as the Newton equation

$$\ddot{x} = -\partial_x V_V(x) \quad (7.5)$$

with the time-dependent potential

$$V_V(x) = -\frac{\alpha}{\sinh^2 x} - \frac{\beta}{\cosh^2 x} + \frac{\gamma e^{2t}}{2} \cosh(2x) + \frac{\delta e^{4t}}{8} \cosh(4x). \quad (7.6)$$

Again, we see that only three parameters among the four are really independent because one of them can be put equal to 1 by a proper shift of t . This equation admits a Hamiltonian structure similar to the previous cases:

$$\dot{x} = \frac{\partial H_V}{\partial p}, \quad \dot{p} = -\frac{\partial H_V}{\partial x}$$

with the classical time-dependent Hamiltonian

$$H_V(p, x) = \frac{p^2}{2} + V_V(x) \quad (7.7)$$

To indicate the dependence on the parameters, we will write $H_V(p, x) = H_V^{(\alpha, \beta, \gamma, \delta)}(p, x)$.

7.2 The zero curvature representation of the P_V equation

The choice of the U - V pair for the P_V equation suitable for our purpose is by no means obvious. We start from a modified version of the U - V pair with rational dependence on the spectral parameter suggested by M.Jimbo and T.Miwa [18] and then show how to transform it to the desired form.

7.2.1 The modified Jimbo-Miwa U - V pair for P_V

Let us consider the system of linear problems

$$\begin{cases} \partial_X \Psi = U(X, t) \Psi \\ \partial_t \Psi = V(X, t) \Psi \end{cases} \quad (7.8)$$

with the matrices

$$U = \begin{pmatrix} \frac{e^{2t}}{2} + \frac{g}{X} - \frac{g + \sigma}{X - 1} & \frac{v}{X} - \frac{w}{X - 1} \\ \frac{v_1}{X} - \frac{w_1}{X - 1} & -\frac{e^{2t}}{2} - \frac{g}{X} + \frac{g + \sigma}{X - 1} \end{pmatrix} \quad (7.9)$$

$$V = \begin{pmatrix} X e^{2t} & 2(v - w) \\ 2(v_1 - w_1) & -X e^{2t} \end{pmatrix} \quad (7.10)$$

and a column 2-component vector Ψ . Here X is the spectral parameter, v, w, v_1, w_1, g are some functions of t to be constrained by the zero curvature condition $\partial_X V - \partial_t U + [V, U] = 0$ and σ is an arbitrary constant. The zero curvature condition yields the system of

differential equations

$$\begin{cases} \dot{g} = 2(vw_1 - wv_1) \\ \dot{v} = -4(v - w)g \\ \dot{w} = -4(v - w)(g + \sigma) + 2we^{2t} \\ \dot{v}_1 = 4(v_1 - w_1)g \\ \dot{w}_1 = 4(v_1 - w_1)(g + \sigma) - 2w_1e^{2t}. \end{cases} \quad (7.11)$$

Combining these equations, one easily finds two integrals:

$$\begin{aligned} vv_1 + g^2 &= \zeta^2, \\ ww_1 + g(g + 2\sigma) &= \xi^2 + 2\xi\sigma, \end{aligned} \quad (7.12)$$

where ζ, ξ are arbitrary constants (the integration constants are expressed in this particular way for later convenience). These formulas allow one to substitute

$$v_1 = \frac{(\zeta - g)(\zeta + g)}{v}, \quad w_1 = \frac{(\xi - g)(2\sigma + \xi + g)}{w}$$

into the first equation of the system (7.11) thus reducing it to the system of three equations for three unknown functions.

Let us introduce the function

$$y = \frac{v}{w}, \quad (7.13)$$

then the first equation of the system (7.11) becomes

$$\dot{g} = 2\left(y^{-1}(g + \xi)(g - \xi) - y(g + \xi + 2\sigma)(g - \xi)\right). \quad (7.14)$$

Writing $\dot{y} = \frac{\dot{v}}{w} - \frac{v\dot{w}}{w^2} = \frac{\dot{v}}{w} - y\frac{\dot{w}}{w}$, we find from the second and third equations of the system (7.11):

$$\dot{y} = 4(y - 1)^2g + 4\sigma y(y - 1) - 2ye^{2t}. \quad (7.15)$$

Plugging

$$g = \frac{\dot{y} + 2ye^{2t}}{4(y - 1)^2} - \frac{\sigma y}{y - 1} \quad (7.16)$$

expressed from this equation in terms of y into (7.14), one obtains, after a relatively long calculation,

$$\ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\dot{y}^2 + 8(y-1)^2\left((\xi + \sigma)^2y - \frac{\zeta^2}{y}\right) + 4(2\sigma-1)e^{2t}y - \frac{2e^{4t}y(y+1)}{y-1} \quad (7.17)$$

which is the P_V equation in the form (7.2) with

$$\alpha = 2(\xi + \sigma)^2, \quad \beta = -2\zeta^2, \quad \gamma = 2\sigma - 1, \quad \delta = -\frac{1}{2}. \quad (7.18)$$

7.2.2 Hyperbolic parametrization

The crucial step of the further construction is a parametrization of the modified Jimbo-Miwa U - V pair (7.9), (7.10) in terms of hyperbolic functions. This parametrization corresponds to the hyperbolic substitution (7.3) for the dependent variable leading to the Calogero form of the P_V equation but does not coincide with it. The next step is a special diagonal gauge transformation which recasts the matrices in the form such that the condition $b_x = 2B$ (2.4) is satisfied.

The required hyperbolic parametrization is achieved by setting

$$X = \cosh^2 x. \quad (7.19)$$

Since this transformation does not depend on t , the general formulas (2.27) simplify. Taking into account the rule $\partial_x = 2 \cosh x \sinh x \partial_X$ by which the derivative ∂_X is transformed, we see that the first linear problem in (7.8) should be changed to $\partial_x \Psi = 2 \cosh x \sinh x U(X(x), t) \Psi$, so the U - V pair in the hyperbolic parametrization acquires the form

$$\tilde{U}(x, t) = \begin{pmatrix} \tilde{a} & 2v \tanh x - 2w \coth x \\ 2v_1 \tanh x - 2w_1 \coth x & -\tilde{a} \end{pmatrix} \quad (7.20)$$

with

$$\tilde{a} = e^{2t} \sinh x \cosh x + 2g \tanh x - (2g + 2\sigma) \coth x$$

and

$$\tilde{V}(x, t) = \begin{pmatrix} e^{2t} \cosh^2 x & 2(v - w) \\ 2(v_1 - w_1) & -e^{2t} \cosh^2 x \end{pmatrix}. \quad (7.21)$$

Here the functions v, w, v_1, w_1, g are the same as in (7.9), (7.10). Clearly, the zero curvature condition yields equation (7.17) with the same constants ζ, ξ as in (7.12). This U - V pair obeys the property $b_x = 2B$.

We deliberately denote the hyperbolic spectral parameter by the same letter x as in the equation of the classical motion (7.4) to stress the fact that it is this variable (x -coordinate of a particle on the line) which is going to be “quantized” in the “quantum” version of the Painlevé-Calogero correspondence in the sense that the momentum $p = \dot{x}$ is going to be replaced by the operator ∂_x .

7.3 Classical-quantum correspondence for P_V

In order to achieve the precise classical-quantum correspondence, one should apply a diagonal gauge transformation. Namely, let us pass to the gauge equivalent U - V pair

$$U = \Omega^{-1} \tilde{U} \Omega - \Omega^{-1} \partial_x \Omega, \quad V = \Omega^{-1} \tilde{V} \Omega - \Omega^{-1} \partial_t \Omega$$

with

$$\Omega = \begin{pmatrix} \left(\frac{e^{-t}(v-w)}{\sinh x \cosh x} \right)^{\frac{1}{2}} & 0 \\ 0 & \left(\frac{e^{-t}(v-w)}{\sinh x \cosh x} \right)^{-\frac{1}{2}} \end{pmatrix}.$$

Explicitly, the U - V pair (7.20), (7.21) transforms into

$$\mathbf{U}(x, t) = \begin{pmatrix} a & \frac{2e^t(v \sinh^2 x - w \cosh^2 x)}{v - w} \\ 2e^{-t}(v - w) \left(\frac{v_1}{\cosh^2 x} - \frac{w_1}{\sinh^2 x} \right) & -a \end{pmatrix} \quad (7.22)$$

with

$$a = e^{2t} \sinh x \cosh x + \left(2g + \frac{1}{2}\right) \tanh x - \left(2g + 2\sigma - \frac{1}{2}\right) \coth x$$

and

$$\mathbf{V}(x, t) = \begin{pmatrix} e^{2t} (\cosh^2 x + \sinh^2 u) - 2\sigma + \frac{1}{2} & e^t \sinh(2x) \\ \frac{4(v - w)(v_1 - w_1)}{e^t \sinh(2x)} & -e^{2t} (\cosh^2 x + \sinh^2 u) + 2\sigma - \frac{1}{2} \end{pmatrix}. \quad (7.23)$$

In principle, the auxiliary functions g, v, w, v_1, w_1 can be excluded from the hyperbolic U - V pair (7.22), (7.23), with the final result being written solely in terms of u and \dot{u} . However, for the purpose of this paper we do not need this form (it will be presented elsewhere). The result of the previous subsections imply that setting

$$y = \frac{v}{w} = \coth^2 u$$

we find from the zero curvature condition for the hyperbolic U - V pair (7.22), (7.23) the P_V equation in the Calogero-Inozemtsev-like form:

$$\ddot{u} = -\frac{4(\xi + \sigma)^2 \cosh u}{\sinh^3 u} + \frac{4\zeta^2 \sinh u}{\cosh^3 u} - (2\sigma - 1)e^{2t} \sinh(2u) + \frac{1}{4} e^{4t} \sinh(4u). \quad (7.24)$$

Note that in this parametrization $b(x) = 2e^t \sinh(x - u) \sinh(x + u)$. For real u this element has just two zeros in the strip $|\operatorname{Im} x| < \pi$ at the points $\pm u$ and the both obey the same equation (7.24).

In order to check the classical-quantum correspondence, one should calculate the potential $U(x, t)$ of the non-stationary Schrödinger equation (2.19). The calculation is straightforward and the result is

$$\begin{aligned} U(x, t) &= \frac{4\zeta^2 - \frac{1}{4}}{2 \cosh^2 x} - \frac{4(\xi + \sigma)^2 - \frac{1}{4}}{2 \sinh^2 x} \\ &\quad - \frac{e^{4t}}{16} \cosh(4x) + \left(\sigma - \frac{1}{2}\right) e^{2t} \cosh(2x) - \tilde{H}, \end{aligned} \quad (7.25)$$

where

$$\tilde{H} = 2(v - w)(v_1 - w_1) - \frac{e^{4t}}{16} - e^{2t} \left(2g + \frac{w}{v - w} + \sigma + \frac{1}{2}\right) + 2\sigma^2. \quad (7.26)$$

The x -dependent part of the potential coincides with the potential (7.4) up to some shifts of the parameters $\alpha \rightarrow \alpha - \frac{1}{8}$, $\beta \rightarrow \beta + \frac{1}{8}$ (see (7.18)). Let us find the x -independent term \tilde{H} and compare it with the classical Hamiltonian for P_V . Using (7.12), we get:

$$2(v - w)(v_1 - w_1) = 2(y - 1) \left(\frac{y - 1}{y} g^2 + 2\sigma g + \frac{\zeta^2}{y} - \xi^2 - 2\xi\sigma \right)$$

and g is given by (7.16). Passing to the hyperbolic parametrization, we have:

$$g = -\frac{\dot{u}}{2} \sinh u \cosh u + \frac{e^{2t}}{2} \sinh^2 u \cosh^2 u - \sigma \cosh^2 u$$

and

$$2(v-w)(v_1-w_1) = \frac{\dot{u}^2}{2} + \frac{e^{4t}}{16} (\cosh(4u) - 1) - \frac{e^{2t}}{2} \sinh(2u) \dot{u} + \frac{2\zeta^2}{\cosh^2 u} - \frac{2(\xi + \sigma)^2}{\sinh^2 u} - 2\sigma^2.$$

Plugging all this into (7.26), we get exactly the classical Hamiltonian $H_V^{(\alpha, \beta, \gamma, \delta)}(\dot{u}, u)$ with the parameters $\alpha, \beta, \gamma, \delta$ given by (7.18):

$$\begin{aligned} \tilde{H} &= \frac{\dot{u}^2}{2} - \frac{2(\xi + \sigma)^2}{\sinh^2 u} + \frac{2\zeta^2}{\cosh^2 u} + \frac{e^{2t}}{2} (2\sigma - 1) \cosh(2u) - \frac{e^{4t}}{16} \cosh(4u) \\ &= H_V^{(\alpha, \beta, \gamma, \delta)}(\dot{u}, u). \end{aligned} \quad (7.27)$$

Summing up, in the case of P_V we have the non-stationary Schrödinger equation

$$\partial_t \psi = \left(H_V^{(\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma, \frac{1}{2})}(\partial_x, x) - H_V^{(\alpha, \beta, \gamma, \frac{1}{2})}(\dot{u}, u) \right) \psi. \quad (7.28)$$

The function

$$\Psi(x, t) = e^{\int^t H_V^{(\alpha, \beta, \gamma, \frac{1}{2})}(\dot{u}, u) dt'} \psi(x, t) \quad (7.29)$$

thus obeys the non-stationary Schrödinger equation

$$\partial_t \Psi = H_V^{(\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma, \frac{1}{2})}(\partial_x, x) \Psi = \left(\frac{1}{2} \partial_x^2 + V_V^{(\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma, \frac{1}{2})}(x, t) \right) \Psi. \quad (7.30)$$

Note that the parameters α, β in the quantum Hamiltonian are shifted by $\pm \frac{1}{8}$.

To conclude, we have two equivalent representations of the P_V equation. One is a classical motion in the time-dependent potential with Hamiltonian (7.7). The coordinate of the particle as a function of time obeys the P_V equation. Another representation is a quantum mechanical particle in the time-dependent potential of the same form with modified coefficients described by a non-stationary Schrödinger equation. The latter simultaneously serves as the linear problem for time evolution associated with the Painlevé equation.

8 Painlevé VI

8.1 The equation

The standard form of the P_{VI} equation is

$$\begin{aligned} \partial_T^2 y &= \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-T} \right) (\partial_T y)^2 - \left(\frac{1}{T} + \frac{1}{T-1} + \frac{1}{y-T} \right) \partial_T y \\ &\quad + \frac{y(y-1)(y-T)}{T^2(T-1)^2} \left(\alpha + \frac{\beta T}{y^2} + \frac{\gamma(T-1)}{(y-1)^2} + \frac{\delta T(T-1)}{(y-T)^2} \right), \end{aligned} \quad (8.1)$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters. Let us perform the change of the variables $(y, T) \rightarrow (x, t)$ given by the formulas [28, 27]

$$y = \frac{\wp(x) - e_1}{e_2 - e_1}, \quad T = \frac{e_3 - e_1}{e_2 - e_1}, \quad (8.2)$$

where $\wp(x) = \wp(x|1, \tau)$ is the Weierstrass \wp -function with periods 1, $\tau = 2\pi it$, and $e_k = \wp(\omega_k)$, $k = 1, 2, 3$, are the values of $\wp(x)$ at the half-periods $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{1}{2}(1 + \tau)$, $\omega_3 = \frac{1}{2}\tau$. It is convenient to set also $\omega_0 = 0$. This change of variables brings the P_{VI} equation to the Newton form

$$\ddot{x} = \sum_{k=0}^3 \nu_k \wp'(x + \omega_k), \quad (8.3)$$

where $\wp'(x) \equiv \partial_x \wp(x)$, and $\nu_0 = \alpha$, $\nu_1 = -\beta$, $\nu_2 = \gamma$, $\nu_3 = -\delta + \frac{1}{2}$. This equation admits the Hamiltonian structure

$$\dot{x} = \frac{\partial H_{VI}}{\partial p}, \quad \dot{p} = -\frac{\partial H_{VI}}{\partial x}$$

with the classical time-dependent Hamiltonian

$$H_{VI}(p, x) = \frac{p^2}{2} + V_{VI}(x), \quad V_{VI}(x) = -\sum_{k=0}^3 \nu_k \wp(x + \omega_k). \quad (8.4)$$

It describes classical motion of a point-like particle in the periodic time-dependent potential. The time dependence is hidden in the second period of the \wp -function:

$$\wp(x) = \wp(x|1, \tau), \quad \tau = 2\pi it. \quad (8.5)$$

To indicate the dependence on the parameters, we will write $H_{VI}(p, x) = H_{VI}^{(\alpha, \beta, \gamma, \delta)}(p, x)$ and $V_{VI}(x, t) = V_{VI}^{(\alpha, \beta, \gamma, \delta)}(x, t)$. The elliptic form of the P_{VI} equation was discussed also in [37, 38].

8.2 The zero curvature representation of the P_{VI} equation

Different versions of the U - V pairs for the P_{VI} equation with spectral parameter on an elliptic curve were found in [39] for the special case of equal constants $\nu_k = \nu$ and in [40] for the general case. However, they appear to be unsuitable for our purpose. Like in the case of the P_V equation, we start from a modified version of the U - V pair with rational dependence on the spectral parameter suggested in [18], then pass to an elliptic parametrization and transform it to the desired form by a gauge transformation.

8.2.1 The modified Jimbo-Miwa U - V pair for P_{VI}

Let us consider the system of linear problems

$$\begin{cases} \partial_X \Psi = U(X, T) \Psi \\ \partial_T \Psi = V(X, T) \Psi \end{cases} \quad (8.6)$$

with the matrices [18]

$$\mathbf{U} = \begin{pmatrix} \frac{g_0 + \xi_0}{X} + \frac{g_1 + \xi_1}{X-1} + \frac{g_2 + \xi_2}{X-T} & -\left(\frac{u_0 g_0}{X} + \frac{u_1 g_1}{X-1} + \frac{u_2 g_2}{X-T}\right) \\ \frac{g_0 + 2\xi_0}{u_0 X} + \frac{g_1 + 2\xi_1}{u_1(X-1)} + \frac{g_2 + 2\xi_2}{u_2(X-T)} & -\left(\frac{g_0 + \xi_0}{X} + \frac{g_1 + \xi_1}{X-1} + \frac{g_2 + \xi_2}{X-T}\right) \end{pmatrix} \quad (8.7)$$

$$\mathbf{V} = \begin{pmatrix} -\frac{g_2 + \xi_2}{X-T} & \frac{u_2 g_2}{X-T} \\ -\frac{g_2 + 2\xi_2}{u_2(X-T)} & \frac{g_2 + \xi_2}{X-T} \end{pmatrix} \quad (8.8)$$

and two-component vector Ψ . Here X is the spectral parameter living on the Riemann sphere, g_i , u_i are some functions of T to be determined from the zero curvature condition $\partial_X \mathbf{V} - \partial_T \mathbf{U} + [\mathbf{V}, \mathbf{U}] = 0$ and ξ_i are arbitrary constants. Below in this section we denote the entries of the matrices \mathbf{U} , \mathbf{V} as $\mathbf{U} = \mathbf{U}(X) = \begin{pmatrix} \mathbf{a}(X) & \mathbf{b}(X) \\ \mathbf{c}(X) & \mathbf{d}(X) \end{pmatrix}$, $\mathbf{V} = \mathbf{V}(X) = \begin{pmatrix} \mathbf{A}(X) & \mathbf{B}(X) \\ \mathbf{C}(X) & \mathbf{D}(X) \end{pmatrix}$ (for brevity, the T -dependence is not indicated explicitly).

The following integrals of motion are immediate consequences of the zero curvature condition:

$$\begin{aligned} g_0 + g_1 + g_2 &= \xi_3 \\ u_0 g_0 + u_1 g_1 + u_2 g_2 &= 0 \\ \frac{g_0 + 2\xi_0}{u_0} + \frac{g_1 + 2\xi_1}{u_1} + \frac{g_2 + 2\xi_2}{u_2} &= 0. \end{aligned} \quad (8.9)$$

Here ξ_3 is an arbitrary constant, the values of the other two integrals are set equal to zero following [18]. The full system of ordinary differential equations for the functions g_i , u_i which follows from the zero curvature condition is explicitly given in Appendix A. Next, let us introduce a function y by representing the right upper entry of the matrix \mathbf{U} in the form

$$\mathbf{b}(X) = \frac{K(X-y)}{X(X-1)(X-T)}, \quad (8.10)$$

where

$$K = T u_0 g_0 + (T-1) u_1 g_1, \quad y = \frac{T u_0 g_0}{K}. \quad (8.11)$$

Note that in terms of K, y we have:

$$u_0 g_0 = \frac{Ky}{T}, \quad u_1 g_1 = -\frac{K(y-1)}{T-1}, \quad u_2 g_2 = \frac{K(y-T)}{T(T-1)}. \quad (8.12)$$

One can see that the zero curvature condition implies the P_{VI} equation (8.1) for the function y with

$$\alpha = 2\left(\xi + \frac{1}{2}\right)^2, \quad \beta = -2\xi_0^2, \quad \gamma = 2\xi_1^2, \quad \delta = \frac{1}{2} - 2\xi_2^2, \quad (8.13)$$

where

$$\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3. \quad (8.14)$$

Some details of the proof are presented in Appendix A.

8.2.2 Elliptic parametrization

The crucial ingredient of the construction is a parametrization of the modified Jimbo-Miwa U - V pair (8.7), (8.8) in terms of elliptic functions. This parametrization corresponds to the elliptic substitution (8.2) for the dependent and independent variables leading to the Calogero form of the P_{VI} equation.

We use the general relations for a change of variables from X, T to x, t of the form $X = X(x, t)$, $T = T(t)$ given in section 2.4. According to these relations, the U - V pair in the variables x, t is

$$\begin{aligned} \tilde{\mathbf{U}}(x, t) &= \frac{\partial X}{\partial x} \mathbf{U}(X(x, t), T(t)) \\ \tilde{\mathbf{V}}(x, t) &= \frac{\partial T}{\partial t} \mathbf{V}(X(x, t), T(t)) + \frac{\partial X}{\partial t} \mathbf{U}(X(x, t), T(t)), \end{aligned} \quad (8.15)$$

where the entries of the matrices \mathbf{U}, \mathbf{V} in the right hand side should be expressed through the new variables x, t (see (8.15)).

We need some formulas which would allow us to make this transformation explicit. It is natural to expect that the change of the time variable is the same as for the P_{VI} equation itself (see the second formula in (8.2)). It turns out that the change of the spectral parameter is also given by the same elliptic function as the one used for the dependent variable in (8.2):

$$X(x, t) = \frac{\wp(x) - e_1}{e_2 - e_1}, \quad T(t) = \frac{e_3 - e_1}{e_2 - e_1} = \left(\frac{\vartheta_3(0|\tau)}{\vartheta_0(0|\tau)} \right)^4. \quad (8.16)$$

Here $\wp(x) = \wp(x|1, \tau)$ and $e_j = \wp(\omega_j|1, \tau)$ depend on the new time variable $t = \frac{\tau}{2\pi i}$ through the second period of the \wp -function. In the last formula, we give the parametrization of T in terms of Jacobi's theta-functions $\vartheta_a(x|\tau)$ (see Appendix B). The arguments leading to equations (8.16) and the derivation are given in Appendix C. Let us also note that the elliptic substitution for the spectral parameter of the form (8.16) was first suggested in [41], where the relation between rational and elliptic forms of the linear problems for the P_{VI} equation was described in terms of modification of the corresponding vector bundles.

Similar to the previously considered cases, we deliberately denote the elliptic spectral parameter by the same letter x as in the equation of the classical motion (8.3) to stress the fact that it is this variable (x -coordinate of a particle on the line) which is going to be “quantized” in the “quantum” version of the Painlevé-Calogero correspondence in the sense that the momentum $p = \dot{x}$ is going to be replaced by the operator ∂_x .

For practical calculations we need some more formulas. First of all, we have

$$X = \frac{\wp(x) - e_1}{e_2 - e_1}, \quad X-1 = \frac{\wp(x) - e_2}{e_2 - e_1}, \quad X-T = \frac{\wp(x) - e_3}{e_2 - e_1}, \quad (8.17)$$

so that the identities

$$\left(\frac{\partial X}{\partial x}\right)^2 = 4(e_2 - e_1)X(X-1)(X-T) \quad (8.18)$$

$$\frac{\partial^2 X}{\partial x^2} = 2(e_2 - e_1)X(X-1)(X-T) \left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-T}\right) \quad (8.19)$$

hold true. (The first one is the differential equation for the \wp -function (B13), the second one is a result of its further differentiating with respect to x .) Next we need the following relations:

$$\begin{aligned} \frac{(e_2 - e_1)T}{X} &= \wp(x + \omega_1) - e_1 \\ -\frac{(e_2 - e_1)(T-1)}{X-1} &= \wp(x + \omega_2) - e_2 \\ \frac{(e_2 - e_1)T(T-1)}{X-T} &= \wp(x + \omega_3) - e_3. \end{aligned} \quad (8.20)$$

At last, let us present formulas for derivatives of the elliptic functions with respect to $t = \frac{\tau}{2\pi i}$. All of them follow from the “heat equation” obeyed by any Jacobi’s theta-function $\vartheta_a(x) = \vartheta_a(x|\tau)$, $a = 0, \dots, 3$:

$$2\partial_t \vartheta_a(x) = \partial_x^2 \vartheta_a(x), \quad t = \frac{\tau}{2\pi i}. \quad (8.21)$$

In particular, we need the following two derivatives:

$$\frac{\partial X}{\partial t} = \frac{\partial X}{\partial x} \frac{\vartheta'_0(x)}{\vartheta_0(x)} \quad (8.22)$$

$$\frac{\partial T}{\partial t} = 2(e_2 - e_1)T(T-1). \quad (8.23)$$

The derivation is given in Appendix B. The formula for $\partial X/\partial t$ first appeared in Takasaki’s paper [29]. Note that differentiating a double-periodic function of x with respect to one of the periods, as in (8.22), we obtain a function which is *not* an elliptic function of x . The second formula is a direct corollary of the definition and (B14). (In fact, since $T = X(\omega_3)$, the second formula follows from the first one).

8.3 Classical-quantum correspondence for P_{VI}

Consider the P_{VI} equation in the Calogero-like form (8.3) for a variable u :

$$\ddot{u} = \sum_{k=0}^3 \nu_k \wp'(u + \omega_k). \quad (8.24)$$

Recall that the variables u, t are connected with the original variables y, T in (8.1) by the formulas (8.16):

$$y = \frac{\wp(u) - e_1}{e_2 - e_1}, \quad T = \frac{e_3 - e_1}{e_2 - e_1} \quad (8.25)$$

and

$$\nu_0 = \alpha = 2\left(\xi + \frac{1}{2}\right)^2, \quad \nu_1 = -\beta = 2\xi_0^2, \quad \nu_2 = \gamma = 2\xi_1^2, \quad \nu_3 = \frac{1}{2} - \delta = 2\xi_2^2. \quad (8.26)$$

This equation is equivalent to the zero curvature condition for the matrices $\tilde{\mathbf{U}}(x, t)$, $\tilde{\mathbf{V}}(x, t)$ given by (8.15) with the elliptic parametrization (8.16).

The next step is a special diagonal gauge transformation $\{\tilde{\mathbf{U}}, \tilde{\mathbf{V}}\} \longrightarrow \{\mathbf{U}, \mathbf{V}\}$ of the form (2.9) that recasts the matrices in the form such that the condition $b_x = 2B$ (2.4) is satisfied. As is shown in Appendix C, the condition that the dependence on x and u in the gauge function ω factorizes is strong enough to fix simultaneously the elliptic substitution for the spectral parameter and the x -dependent part of ω . The latter is found in the form

$$\omega^2 = \frac{\wp'(x) \vartheta_0^2(x)}{2(\wp(x) - e_3)} \rho^2(t), \quad (8.27)$$

where $\rho(t)$ is some (yet unknown) function of t only (see (C14)). The function ρ is to be determined at the very end from the condition that the x -independent part of the potential in the non-stationary Schrödinger equation be equal to the classical Hamiltonian $H_{\text{VI}}(\dot{u}, u)$.

The detailed derivation of (8.27) is given in Appendix C. Here we can only say that if this expression is known, then it is an easy exercise to check that the x -derivative of the right upper element b of the matrix \mathbf{U} ,

$$b = \omega^2 \tilde{b} = 2K(e_2 - e_1) \rho^2(t) \vartheta_0^2(x) \frac{\wp(x) - \wp(u)}{\wp(x) - e_3}, \quad (8.28)$$

appears to be equal to $2B = 2\omega^2 \tilde{B}$. Therefore, in this gauge the non-stationary Schrödinger equation of the form (2.19) does hold. It remains to find the potential $U(x, t) = \frac{1}{2}(ad - bc - a_x + 2A)$. Taking into account that $a = \tilde{a} + \partial_x \log \omega$, $A = \tilde{A} + \partial_t \log \omega$ and $\tilde{a} = \frac{\partial X}{\partial x} \mathbf{a}$, $\tilde{A} = \frac{\partial T}{\partial t} \mathbf{A} + \frac{\partial X}{\partial x} \frac{\vartheta_0'(x)}{\vartheta_0(x)} \mathbf{a}$, one can represent it as a sum of three terms: $U = U_1 + U_2 + U_3$, where

$$\begin{aligned} U_1 &= \frac{1}{2} \left(\frac{\partial X}{\partial x} \right)^2 \det \mathbf{U}(X) = \frac{1}{2} \left(\frac{\partial X}{\partial x} \right)^2 (ad - bc) \\ U_2 &= -\frac{1}{2} \left[\frac{\partial^2 X}{\partial x^2} \mathbf{a} + \left(\frac{\partial X}{\partial x} \right)^2 \mathbf{a}_X \right] + \frac{\partial T}{\partial t} \mathbf{A} + \frac{\partial X}{\partial x} \frac{\vartheta_0'(x)}{\vartheta_0(x)} \mathbf{a} - \frac{\partial X}{\partial x} \mathbf{a} \partial_x \log \omega \\ U_3 &= -\frac{1}{2} (\partial_x \log \omega)^2 - \frac{1}{2} \partial_x^2 \log \omega + \partial_t \log \omega. \end{aligned}$$

For the purpose of this paper we do not need the explicit form of the matrices $\mathbf{U}(x, t)$, $\mathbf{V}(x, t)$ in the elliptic parametrization (this will be presented elsewhere). Technically, it is convenient to make the calculations using the original variables X, T where possible and pass to the elliptic parametrization at the very end. That is why we have expressed the right hand sides in terms of the matrices \mathbf{U}, \mathbf{V} with rational dependence on the spectral parameter X .

The calculation of the X -dependent part of U_1 is relatively easy. The result is:

$$\begin{aligned}
U_1 &= 2(e_2 - e_1) \left[-\xi^2 X - \frac{T\xi_0^2}{X} + \frac{(T-1)\xi_1^2}{X-1} - \frac{T(T-1)\xi_2^2}{X-T} \right] + U_{1,0} \\
&= -2\xi^2(\wp(x) - e_1) - 2\xi_0^2(\wp(x + \omega_1) - e_1) - 2\xi_1^2(\wp(x + \omega_2) - e_2) \\
&\quad - 2\xi_2^2(\wp(x + \omega_3) - e_3) + U_{1,0}.
\end{aligned} \tag{8.29}$$

The passage to the elliptic functions is done according to formulas (8.20). The X -independent part, $U_{1,0}$, is

$$U_{1,0} = 2\xi(e_2 - e_1) \left[(T+1)(g_0 + \xi_0) + (T-1)(g_1 + \xi_1) - (T-1)(g_2 + \xi_2) \right]. \tag{8.30}$$

Using formulas from Appendix A, we get:

$$\begin{aligned}
U_{1,0} &= \left(2\xi^2 - \frac{1}{2} \right) (e_2 - e_1) y + \frac{2(e_2 - e_1)T}{y} \left(-\frac{(T-1)^2 y_T^2}{4} + \xi_0^2 \right) \\
&+ \frac{2(e_2 - e_1)(T-1)}{y-1} \left(\frac{T^2 y_T^2}{4} - \xi_1^2 \right) + \frac{2(e_2 - e_1)T(T-1)}{y-T} \left(-\frac{(y_T-1)^2}{4} + \xi_2^2 \right) \\
&- \frac{(e_2 - e_1)(T-1)}{2}.
\end{aligned} \tag{8.31}$$

The T -derivative of y in the elliptic parametrization reads

$$y_T = \frac{1}{2T(T-1)} \frac{\wp'(u)}{(e_2 - e_1)^2} \left(\dot{u} + \frac{\wp'_0(u)}{\wp_0(u)} \right). \tag{8.32}$$

Plugging it to the right hand side of (8.31) and using formulas (8.20) (now with y, u instead of X, x), we obtain:

$$\begin{aligned}
U_{1,0} &= \left(2\xi^2 - \frac{1}{2} \right) (\wp(u) - e_1) + 2\xi_0^2(\wp(u + \omega_1) - e_1) \\
&+ 2\xi_1^2(\wp(u + \omega_2) - e_2) + \left(2\xi_2^2 - \frac{1}{2} \right) (\wp(u + \omega_3) - e_3) \\
&- \frac{1}{2} \left(\dot{u} + \frac{\wp'_0(u)}{\wp_0(u)} \right)^2 + \frac{\wp'(u)}{2(e_2 - e_1)(y-T)} \left(\dot{u} + \frac{\wp'_0(u)}{\wp_0(u)} \right) - \frac{e_3 - e_2}{2}.
\end{aligned} \tag{8.33}$$

The unwanted terms in the last line can be transformed to logarithmic t -derivatives using the formulas

$$\partial_t \log \wp_0(u) = \dot{u} \frac{\wp'_0(u)}{\wp_0(u)} + \frac{1}{2} \left(\frac{\wp'_0(u)}{\wp_0(u)} \right)^2 - \frac{1}{2} \wp(u + \omega_3) - \eta \tag{8.34}$$

$$\frac{1}{2} \partial_t \log(y-T) = \frac{\wp'(u)}{2(e_2 - e_1)(y-T)} \left(\dot{u} + \frac{\wp'_0(u)}{\wp_0(u)} \right) - \wp(u + \omega_3) + e_3 \tag{8.35}$$

(and thus they can be eliminated by a proper choice of the function $\rho(t)$, see below). Taking this into account, we obtain $U_{1,0}$ in the form

$$\begin{aligned}
U_{1,0} &= \left(2\xi^2 - \frac{1}{2}\right)(\wp(u) - e_1) + 2\xi_0^2(\wp(u + \omega_1) - e_1) \\
&\quad + 2\xi_1^2(\wp(u + \omega_2) - e_2) + 2\xi_2^2(\wp(u + \omega_3) - e_3) \\
&\quad - \frac{\dot{u}^2}{2} + \partial_t \log \frac{(y - T)^{1/2}}{\vartheta_0(u)} - e_3 + \frac{e_2}{2} - \eta.
\end{aligned} \tag{8.36}$$

For the calculation of U_2 we prepare the formulas

$$\begin{aligned}
\frac{\partial X}{\partial x} \partial_x \log \omega &= \frac{1}{2} \frac{\partial X}{\partial x} \partial_x \log \left(\frac{\wp'(x)}{e_2 - e_1} \frac{e_2 - e_1}{\wp(x) - e_3} \right) + \frac{\partial X}{\partial x} \frac{\vartheta'_0(x)}{\vartheta_0(x)} \\
&= \frac{1}{2} \left[\frac{\partial^2 X}{\partial x^2} - \frac{1}{X - T} \left(\frac{\partial X}{\partial x} \right)^2 \right] + \frac{\partial X}{\partial x} \frac{\vartheta'_0(x)}{\vartheta_0(x)}
\end{aligned}$$

$$\frac{\partial^2 X}{\partial x^2} \mathbf{a} + \frac{1}{2} \left(\frac{\partial X}{\partial x} \right)^2 \mathbf{a}_X = 2(e_2 - e_1) [2\xi X - (T + 1)(g_0 + \xi_0) - T(g_1 + \xi_1) - (g_2 + \xi_2)]$$

and

$$\frac{1}{2} \left(\frac{\partial X}{\partial x} \right)^2 \frac{\mathbf{a}}{X - T} + \frac{\partial T}{\partial t} \mathbf{A} = 2(e_2 - e_1) [\xi X - (g_0 + \xi_0) + (T - 1)(g_2 + \xi_2)].$$

The calculation gives the following simple result:

$$U_2 = -2\xi(\wp(x) - e_3). \tag{8.37}$$

At last, let us find U_3 . We have:

$$\begin{aligned}
\partial_t \log \omega &= \frac{1}{2} \partial_t \log \left(\frac{\wp'(x)}{\wp(x) - e_3} \right) + \partial_t \log(\rho \vartheta_0(x)) \\
&= \frac{1}{2} \partial_t \log \frac{\partial X}{\partial x} - \frac{1}{2} \partial_t \log(X - T) + \partial_t \log(\rho \vartheta_0(x)) \\
&= \frac{1}{2} \left(\frac{\partial X}{\partial x} \right)^{-1} \frac{\partial^2 X}{\partial x^2} + \frac{1/2}{X - T} \left(\frac{\partial T}{\partial t} - \frac{\partial X}{\partial t} \right) + \partial_t \log \vartheta_0(x) + \partial_t \log \rho \\
&= \frac{1}{2} \partial_x \log \left(\frac{\wp'(x)}{\wp(x) - e_3} \right) \partial_x \log \vartheta_0(x) + \frac{1}{2} (\partial_x \log \vartheta_0(x))^2 + \partial_t \log \rho - e_3 - 2\eta,
\end{aligned}$$

where $\eta = -\frac{1}{6} \frac{\vartheta_1'''(0)}{\vartheta_1(0)}$. When passing to the last line we have used the heat equation (8.21)

and the relation $\partial_x^2 \log \vartheta_0(x) = -\wp(x + \omega_3) - 2\eta$. Combining the different contributions to U_3 and passing to the elliptic parametrization, we find:

$$U_3 = -\frac{1}{8} (3\wp(x) - \wp(x + \omega_1) - \wp(x + \omega_2) - \wp(x + \omega_3)) - \frac{e_3}{2} - \eta + \partial_t \log \rho. \tag{8.38}$$

Using the formulas given above and equation (A13), we obtain the potential in the form

$$\begin{aligned}
U(x, t) = & - \left(2 \left(\xi + \frac{1}{2} \right)^2 - \frac{1}{8} \right) \wp(x) - \left(2\xi_0^2 - \frac{1}{8} \right) \wp(x + \omega_1) \\
& - \left(2\xi_1^2 - \frac{1}{8} \right) \wp(x + \omega_2) - \left(2\xi_2^2 - \frac{1}{8} \right) \wp(x + \omega_3) - \tilde{H},
\end{aligned} \tag{8.39}$$

where

$$\tilde{H} = \frac{\dot{u}^2}{2} - \sum_{k=0}^3 \nu_k \wp(u + \omega_k) + \frac{1}{2} \partial_t \log \left(\frac{\vartheta_0^2(u) (\vartheta_1'(0))^{\frac{8}{3}}}{(y - T) K(T) \vartheta_0^6(0) \rho^2(t)} \right)$$

with the same ν_k as in (8.26). Using the identities from Appendix B one can express $y - T$ in terms of the theta-functions:

$$\frac{1}{y - T} = \frac{e_2 - e_1}{\wp(u) - e_3} = - \frac{\pi^2 \vartheta_0^6(0) \vartheta_1^2(u)}{(\vartheta_1'(0))^2 \vartheta_0^2(u)}.$$

Therefore, choosing

$$\rho(t) = \frac{(\vartheta_1'(0))^{\frac{1}{3}} \vartheta_1(u)}{\sqrt{K(T)}},$$

we see that $\tilde{H} = H_{\text{VI}}^{(\alpha, \beta, \gamma, \delta)}(\dot{u}, u)$ with the same parameters as in (8.13). With this choice of ρ , the gauge function ω (8.27) acquires the form

$$\omega^2 = \frac{(\vartheta_1'(0))^{\frac{5}{3}} \vartheta_0(0)}{\vartheta_2(0) \vartheta_3(0)} \frac{\vartheta_2(x) \vartheta_3(x) \vartheta_0(x) \vartheta_1^2(u)}{\vartheta_1(x) K(T)}. \tag{8.40}$$

Finally, we conclude that the classical-quantum correspondence does work for the P_{VI} equation. The non-stationary Schrödinger equation is

$$\partial_t \psi = \left[H_{\text{VI}}^{(\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma - \frac{1}{8}, \delta + \frac{1}{8})}(\partial_x, x) - H_{\text{VI}}^{(\alpha, \beta, \gamma, \delta)}(\dot{u}, u) \right] \psi, \tag{8.41}$$

where

$$H_{\text{VI}}^{(\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma - \frac{1}{8}, \delta + \frac{1}{8})}(\partial_x, x) = \frac{1}{2} \partial_x^2 - \sum_{k=0}^3 \left(\nu_k - \frac{1}{8} \right) \wp(x + \omega_k)$$

and the parameters ν_k are connected with $\alpha, \beta, \gamma, \delta$ as in (8.26). The function

$$\Psi(x, t) = e^{\int^t H_{\text{VI}}^{(\alpha, \beta, \gamma, \delta)}(\dot{u}, u) dt'} \psi(x, t) \tag{8.42}$$

thus obeys the non-stationary Schrödinger equation

$$\partial_t \Psi = H_{\text{VI}}^{(\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma - \frac{1}{8}, \delta + \frac{1}{8})}(\partial_x, x) \Psi = \left(\frac{1}{2} \partial_x^2 + V_{\text{VI}}^{(\alpha - \frac{1}{8}, \beta + \frac{1}{8}, \gamma - \frac{1}{8}, \delta + \frac{1}{8})}(x, t) \right) \Psi. \tag{8.43}$$

Note that in the quantum part all the parameters undergo shifts by $\pm \frac{1}{8}$. In terms of the parameters ν_k (see (8.3), (8.26)) the shifts are $\nu_k \rightarrow \nu_k - \frac{1}{8}$, $k = 0, \dots, 3$. In particular, if all ν_k are equal to each other, $\nu_k = \nu$, then we obtain the non-stationary Lamé equation

$$\partial_t \Psi = \left(\frac{1}{2} \partial_x^2 - 4\tilde{\nu} \wp(2x \mid 1, 2\pi i t) \right) \Psi, \quad \tilde{\nu} = \nu - \frac{1}{8}. \tag{8.44}$$

(the identity $\sum_{k=0}^3 \wp(x + \omega_k) = 4\wp(2x)$ has been used). We remark that the non-stationary Lamé equation in connection with the P_{VI} equation (and with the 8-vertex model) was discussed in [42]. Recently, the non-stationary Lamé equation has appeared [43, 44] in the context of the AGT conjecture.

To summarize, similar to the other cases, we have two equivalent representations of the P_{VI} equation. One is a classical motion in the time-dependent periodic potential with Hamiltonian (8.4). The coordinate of the particle as a function of time obeys the P_{VI} equation. Another representation is a quantum mechanical particle in the time-dependent potential of the same form with modified coefficients described by a non-stationary Schrödinger equation. The latter simultaneously serves as the linear problem for time evolution associated with the Painlevé equation.

9 Concluding remarks

We have shown that for each Painlevé equation written in the “Calogero form” $\ddot{u} = -\partial_u V(u, t)$ with a time-dependent potential $V(x, t)$, the associated linear problems can be represented as

$$\begin{cases} \left(\frac{1}{2} \partial_x^2 - \frac{1}{2} (\partial_x \log b(x, t)) \partial_x + \tilde{W}(x, t) \right) \Psi = E \Psi \\ \partial_t \Psi = \left(\frac{1}{2} \partial_x^2 + \tilde{V}(x, t) \right) \Psi, \end{cases} \quad (9.1)$$

The second equation is the *non-stationary Schrödinger equation in imaginary time with the potential $\tilde{V}(x, t)$ that has the same form as the classical potential for the Painlevé equation* (with possibly modified parameters). The potential in the first equation is

$$\tilde{W}(x, t) = \tilde{V}(x, t) - \frac{\partial_t b(x, t)}{2b(x, t)} + \frac{\partial_x^2 b(x, t)}{4b(x, t)}$$

and the eigenvalue E is the value of the classical Hamiltonian $H(\dot{u}, u)$ for the Painlevé equation in the Calogero form (with the opposite sign):

$$E = -H(\dot{u}, u) = -\frac{\dot{u}^2}{2} - V(u, t).$$

These equations have been derived from the 2×2 matrix linear problems (1.3) with the matrices $\mathbf{U}(x, t)$, $\mathbf{V}(x, t)$ of the special form, with the function $b(x, t)$ being the right upper entry of the matrix $\mathbf{U}(x, t)$. The second equation of the system (9.1) describes isomonodromic deformations of the first one and their compatibility implies the Painlevé equation (in the Calogero form) for the function $u = u(t)$ defined implicitly as zero of the function $b(x, t)$: $b(u(t), t) = 0$.

In short, the conclusion is that linearization of the Painlevé equation, i.e., passing to the linear problem, is equivalent to its quantization. The imaginary time suggests an interpretation in terms of the Fokker-Planck equation for a stochastic process.

Here a remark is in order. On the one hand, the Painlevé equation is obtained as a compatibility condition for the pair of equations (9.1). However, on the other hand, the second equation alone is already enough to encode the full information about the Painlevé equation. Indeed, it describes a quantum mechanical particle on the line in the time-dependent potential corresponding to the Painlevé equation. Therefore, the Painlevé equation itself should emerge in the classical limit.

In the papers [45, 46] the Knizhnik-Zamolodchikov system of equations was treated as a natural quantization of isomonodromic deformations. It would be very interesting to understand our results in these terms.

At last, we would like to point out that another sort of classical-quantum correspondence for Painlevé equations was established in the work [47]. Namely, it was shown that each equation from the Painlevé list could be regarded as a “classical analog” of a linear ordinary differential equation of the Heun class in the sense that the second-order differential operator $\hat{L}(\partial_x, x)$ involved in the latter, after a properly taken classical limit, coincides with the polynomial classical Hamiltonian for the Painlevé equation. (In other words, the Euler-Lagrange equation corresponding to the symbol $L(p, q)$ of the linear differential operator \hat{L} is just the Painlevé equation.) Similarly to our approach, in this classical/quantum mechanical interpretation, the time variable T has the meaning of the deformation parameter. However, the important difference is that [47] deals with *stationary* Schrödinger-like equation with coefficients depending on T . It seems to us that the construction elaborated in the present paper and a similar construction suggested previously in [30] are more appropriate because the Painlevé equations are essentially non-autonomous systems and it is really natural to associate *non-stationary* Schrödinger equations with them.

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Appendix A

In this appendix we present some details of the derivation of the P_{VI} equation from the zero curvature condition for matrices (8.7), (8.8). We use the notation introduced in the main text.

First of all, let us write down the differential equations for the functions g_i , u_i that follow from the zero curvature condition. The full system of equations reads

$$\begin{aligned}
T\partial_T(u_0g_0) &= 2u_0g_0(g_0 + g_2 + \xi_0 + \xi_2) + 2u_1g_1(g_0 + \xi_0) \\
(T-1)T\partial_T(u_1g_1) &= 2u_1g_1(g_1 + g_2 + \xi_1 + \xi_2) + 2u_0g_0(g_1 + \xi_1) \\
T\partial_Tg_0 &= \frac{u_0}{u_2}g_0(g_2 + 2\xi_2) - \frac{u_2}{u_0}g_2(g_0 + 2\xi_0) \\
(T-1)\partial_Tg_1 &= \frac{u_1}{u_2}g_1(g_2 + 2\xi_2) - \frac{u_2}{u_1}g_2(g_1 + 2\xi_1) \\
T\partial_T\left(\frac{g_0 + 2\xi_0}{u_0}\right) &= \frac{2}{u_0}(g_0 + 2\xi_0)(g_2 + \xi_2) - \frac{2}{u_2}(g_2 + 2\xi_2)(g_0 + \xi_0) \\
(T-1)\partial_T\left(\frac{g_1 + 2\xi_1}{u_1}\right) &= \frac{2}{u_1}(g_1 + 2\xi_1)(g_2 + \xi_2) - \frac{2}{u_2}(g_2 + 2\xi_2)(g_1 + \xi_1)
\end{aligned} \tag{A1}$$

along with the integrated relations (8.9). However, a direct derivation of the P_{VI} equation from this system is not the easiest way. Below we give a short-cut which closely follows the derivation outlined in [18].

Along with the function y defined by (8.10) let us also introduce the function

$$z = \mathbf{a}(y) = \frac{g_0 + \xi_0}{y} + \frac{g_1 + \xi_1}{y-1} + \frac{g_2 + \xi_2}{y-T}. \tag{A2}$$

Then, from the fact that the total T -derivative of $\mathbf{b}(y)$ is zero, we write, using the zero curvature equations in the form (2.3): $0 = d\mathbf{b}(y)/dT = \mathbf{b}_X(y)y_T + \mathbf{b}_T(y) = \mathbf{b}_T(y)y_T + \mathbf{B}_X(y) - 2z\mathbf{B}(y) = 0$, where $y_T \equiv dy/dT$. Expressing $\mathbf{b}_X(y)$, etc in terms of the functions K and y (see (8.11)), we obtain:

$$y_T = \frac{y(y-1)(y-T)}{T(T-1)} \left(2z + \frac{1}{y-T} \right). \tag{A3}$$

Combining the integrals of motion (8.9) with the definition of z , and using formulas (8.12), we can write the system of equations

$$\begin{cases} g_0 + g_1 + g_2 = \xi_3 \\ \frac{g_0 + \xi_0}{y} + \frac{g_1 + \xi_1}{y-1} + \frac{g_2 + \xi_2}{y-T} = z \\ \frac{Tg_0(g_0 + 2\xi_0)}{y} - \frac{(T-1)g_1(g_1 + 2\xi_1)}{y-1} + \frac{T(T-1)g_2(g_2 + 2\xi_2)}{y-T} = 0 \end{cases} \tag{A4}$$

for the three functions g_i which can be solved as

$$\begin{aligned}
g_0 = & -\frac{y}{2\xi T} \left[y(y-1)(y-T)\tilde{z}^2 - 2\left(\xi_3(y-1)(y-T) - \xi_1(y-T) - \xi_2T(y-1) \right)\tilde{z} \right. \\
& \left. + \xi_3\left(\xi_3(y-1) - (2\xi_2 + \xi_3)T - 2\xi_1 \right) \right]
\end{aligned} \tag{A5}$$

$$g_1 = \frac{y-1}{2\xi(T-1)} \left[y(y-1)(y-T)\tilde{z}^2 - 2(\xi_3 y(y-T) + \xi_0(y-T) - \xi_2(T-1)y)\tilde{z} \right. \\ \left. + \xi_3(\xi_3(y-1) - (2\xi_2 + \xi_3)(T-1) + 2\xi_0 + \xi_3) \right] \quad (\text{A6})$$

$$g_2 = -\frac{y-T}{2\xi T(T-1)} \left[y(y-1)(y-T)\tilde{z}^2 - 2(\xi_3 y(y-1) + \xi_0 T(y-1) + \xi_1(T-1)y)\tilde{z} \right. \\ \left. + \xi_3(\xi_3(y-1) + (2\xi_0 + \xi_3)T + 2\xi_1(T-1)) \right], \quad (\text{A7})$$

where

$$\tilde{z} = z - \frac{\xi_0}{y} - \frac{\xi_1}{y-1} - \frac{\xi_2}{y-T} = \frac{g_0}{y} + \frac{g_1}{y-1} + \frac{g_2}{y-T} \quad (\text{A8})$$

and $\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3$. In order to find a more explicit representation, we notice that the functions g_0/y , $g_1/(y-1)$ and $g_2/(y-T)$ are rational functions of the variable y with first order poles at 0, 1, T and ∞ . Calculating the residues, one can write them in the explicit form:

$$2\xi \frac{g_0}{y} = -\frac{(\xi + \frac{1}{2})^2}{T} y - \frac{G_0}{T} - \frac{1}{y} \left[\frac{(T-1)^2}{4} y_T^2 - \xi(T-1)y_T + \xi_0(2\xi - \xi_0) \right] \\ + \frac{T-1}{T(y-1)} \left[\frac{T^2}{4} y_T^2 - \xi_1^2 \right] - \frac{T-1}{y-T} \left[\frac{1}{4} (y_T-1)^2 - \xi_2^2 \right] \quad (\text{A9})$$

$$2\xi \frac{g_1}{y-1} = \frac{(\xi + \frac{1}{2})^2}{T-1} y + \frac{G_1}{T-1} + \frac{T}{(T-1)y} \left[\frac{(T-1)^2}{4} y_T^2 - \xi_0^2 \right] \\ + \frac{1}{y-1} \left[-\frac{T^2}{4} y_T^2 - \xi T y_T + \xi_1(\xi_1 - 2\xi) \right] + \frac{T}{y-T} \left[\frac{1}{4} (y_T-1)^2 - \xi_2^2 \right] \quad (\text{A10})$$

$$2\xi \frac{g_2}{y-T} = -\frac{(\xi + \frac{1}{2})^2}{T(T-1)} y - \frac{G_2}{T(T-1)} - \frac{1}{(T-1)y} \left[\frac{(T-1)^2}{4} y_T^2 - \xi_0^2 \right] \\ + \frac{1}{T(y-1)} \left[\frac{T^2}{4} y_T^2 - \xi_1^2 \right] - \frac{1}{y-T} \left[\frac{1}{4} (y_T-1)^2 - \xi(y_T-1) + \xi_2(2\xi - \xi_2) \right] \quad (\text{A11})$$

and

$$G_0 = \frac{T-1}{4} - \xi(\xi T + \xi + 1) \\ G_1 = \frac{T-1}{4} - \xi^2(T-1) \\ G_2 = \frac{T-1}{4} + \xi(\xi + 1)(T-1). \quad (\text{A12})$$

The next step is to express the T -derivative of the function z in terms of the functions g_i and y . For that purpose, we write $z_T = \mathbf{a}_X(y)y_T + \mathbf{a}_T(y)$ and use the zero curvature equation $\mathbf{a}_T(X) - \mathbf{A}_X(X) + \mathbf{b}(X)\mathbf{C}(X) - \mathbf{c}(X)\mathbf{B}(X) = 0$ to obtain

$$\begin{aligned} z_T &= \mathbf{a}_X(y)y_T + \mathbf{A}_X(y) + \mathbf{c}(y)\mathbf{B}(y) \\ &= -\left(\frac{g_0 + \xi_0}{y^2} + \frac{g_1 + \xi_1}{(y-1)^2} + \frac{g_2 + \xi_2}{(y-T)^2}\right)y_T + \frac{g_2 + \xi_2}{(y-T)^2} \\ &\quad + \frac{1}{T(T-1)}\left(\frac{Tg_0(g_0 + 2\xi_0)}{y^2} - \frac{(T-1)g_1(g_1 + 2\xi_1)}{(y-1)^2} + \frac{T(T-1)g_2(g_2 + 2\xi_2)}{(y-T)^2}\right). \end{aligned}$$

It remains to express z_T in terms of y , y_T , y_{TT} with the help of (A3) and to plug the explicit form of the functions g_i given by equations (A9)–(A11). After a long calculation, one obtains the P_{VI} equation (8.1) with the parameters (8.13).

In establishing the classical-quantum correspondence we also need the T -derivative of the function $K(T)$. A straightforward calculation, which uses formulas (8.12) and the first two equations of the system (A1), yields:

$$\partial_T \log K = -(2\xi + 1) \frac{y - T}{T(T-1)}. \quad (\text{A13})$$

Appendix B

Theta-functions, Weierstrass \wp -function and other useful functions

Theta-functions. The Jacobi's theta-functions $\vartheta_a(z) = \vartheta_a(z|\tau)$, $a = 0, 1, 2, 3$, are defined by the formulas

$$\begin{aligned} \vartheta_1(z) &= -\sum_{k \in \mathbb{Z}} \exp\left(\pi i \tau \left(k + \frac{1}{2}\right)^2 + 2\pi i \left(z + \frac{1}{2}\right) \left(k + \frac{1}{2}\right)\right), \\ \vartheta_2(z) &= \sum_{k \in \mathbb{Z}} \exp\left(\pi i \tau \left(k + \frac{1}{2}\right)^2 + 2\pi i z \left(k + \frac{1}{2}\right)\right), \\ \vartheta_3(z) &= \sum_{k \in \mathbb{Z}} \exp\left(\pi i \tau k^2 + 2\pi i z k\right), \\ \vartheta_0(z) &= \sum_{k \in \mathbb{Z}} \exp\left(\pi i \tau k^2 + 2\pi i \left(z + \frac{1}{2}\right) k\right), \end{aligned} \quad (\text{B1})$$

where τ is a complex parameter (the modular parameter) such that $\text{Im } \tau > 0$. The function $\vartheta_1(z)$ is odd, the other three functions are even. The infinite product representation for the $\vartheta_1(z)$ reads:

$$\vartheta_1(z) = i \exp\left(\frac{i\pi\tau}{4} - i\pi z\right) \prod_{k=1}^{\infty} \left(1 - e^{2\pi i k \tau}\right) \left(1 - e^{2\pi i ((k-1)\tau + z)}\right) \left(1 - e^{2\pi i (k\tau - z)}\right). \quad (\text{B2})$$

In order to unify some formulas given below, it is convenient to understand the index a modulo 4, i.e., to identify $\vartheta_a(z) \equiv \vartheta_{a+4}(z)$. Set

$$\omega_0 = 0, \quad \omega_1 = \frac{1}{2}, \quad \omega_2 = \frac{1+\tau}{2}, \quad \omega_3 = \frac{\tau}{2},$$

then the function $\vartheta_a(z)$ has simple zeros at the points of the lattice $\omega_{a-1} + \mathbb{Z} + \mathbb{Z}\tau$ (here $\omega_a \equiv \omega_{a+4}$). The theta-functions have the following quasi-periodic properties under shifts by 1 and τ :

$$\begin{aligned} \vartheta_a(z+1) &= e^{\pi i(1+\partial_\tau \omega_{a-1})} \vartheta_a(z) \\ \vartheta_a(z+\tau) &= e^{\pi i(a+\partial_\tau \omega_{a-1})} e^{-\pi i\tau-2\pi iz} \vartheta_a(z). \end{aligned} \tag{B3}$$

Shifts by the half-periods relate the different theta-functions to each other:

$$\vartheta_1(z+\omega_1) = \vartheta_2(z), \quad \vartheta_3(z+\omega_1) = \vartheta_0(z), \tag{B4}$$

$$\vartheta_1(z+\omega_2) = e^{-\frac{\pi i\tau}{4}-\pi iz} \vartheta_3(z), \quad \vartheta_2(z+\omega_2) = -ie^{-\frac{\pi i\tau}{4}-\pi iz} \vartheta_0(z) \tag{B5}$$

$$\vartheta_1(z+\omega_3) = ie^{-\frac{\pi i\tau}{4}-\pi iz} \vartheta_0(z), \quad \vartheta_2(z+\omega_3) = e^{-\frac{\pi i\tau}{4}-\pi iz} \vartheta_3(z). \tag{B6}$$

Weierstrass \wp -function. The Weierstrass \wp -function can be defined by the formula

$$\wp(z) = -\partial_z^2 \log \vartheta_1(z) - 2\eta, \tag{B7}$$

where

$$\eta = -\frac{1}{6} \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = -\frac{2\pi i}{3} \partial_\tau \log \vartheta_1'(0|\tau). \tag{B8}$$

The function $\wp(z)$ is double-periodic with periods $2\omega_1 = 1$, $2\omega_3 = \tau$, $\wp(z+M+N\tau) = \wp(z)$, $M, N \in \mathbb{Z}$, and has second order poles at the origin (and at all the points $M+N\tau$ with integer M, N). The derivative of the \wp -function is given by

$$\wp'(z) = -\frac{2(\vartheta_1'(0))^3}{\vartheta_2(0)\vartheta_3(0)\vartheta_0(0)} \frac{\vartheta_2(z)\vartheta_3(z)\vartheta_0(z)}{\vartheta_1^3(z)}. \tag{B9}$$

The values of the \wp -function at the half-periods, ω_k ,

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_2), \quad e_3 = \wp(\omega_3) \tag{B10}$$

play a special role. The sum of the numbers e_k is zero: $e_1 + e_2 + e_3 = 0$. The differences $e_j - e_k$ can be represented in terms of the values of the theta-functions at $z = 0$ (theta-constants) in two different ways:

$$\begin{aligned} e_1 - e_2 &= \pi^2 \vartheta_0^4(0) = 4\pi i \partial_\tau \log \frac{\vartheta_3(0)}{\vartheta_2(0)} \\ e_1 - e_3 &= \pi^2 \vartheta_3^4(0) = 4\pi i \partial_\tau \log \frac{\vartheta_0(0)}{\vartheta_2(0)} \\ e_2 - e_3 &= \pi^2 \vartheta_2^4(0) = 4\pi i \partial_\tau \log \frac{\vartheta_0(0)}{\vartheta_3(0)}. \end{aligned} \tag{B11}$$

The second representation is a consequence of the heat equation (B30) (see below). Its another consequence is a representation of the e_k 's themselves as logarithmic τ -derivatives of the theta-constants:

$$e_k = 4\pi i \partial_\tau \left(\frac{1}{3} \log \vartheta'_1(0) - \log \vartheta_{k+1}(0) \right). \quad (\text{B12})$$

Using the first equalities in (B11) and the heat equation, the τ -derivatives of the differences $e_j - e_k$ can be expressed through the e_k 's and η as follows:

$$\pi i \partial_\tau \log(e_j - e_k) = -e_l - 2\eta. \quad (\text{B13})$$

Here $\{jkl\}$ stands for any cyclic permutation of $\{123\}$. Subtracting two such equations, we also get

$$\pi i \partial_\tau \log \frac{e_j - e_k}{e_l - e_k} = e_j - e_l. \quad (\text{B14})$$

The \wp -function obeys the differential equation

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \quad (\text{B15})$$

We also mention the formulae

$$\wp(z) - e_k = \frac{(\vartheta'_1(0))^2}{\vartheta_{k+1}^2(0)} \frac{\vartheta_{k+1}^2(z)}{\vartheta_1^2(z)}. \quad (\text{B16})$$

Eisenstein functions and Φ -function. Sometimes it is convenient to use the Eisenstein functions

$$E_1(z) = \partial_z \log \vartheta_1(z), \quad E_2(z) = -\partial_z E_1(z) = -\partial_z^2 \log \vartheta_1(z) = \wp(z) + 2\eta. \quad (\text{B17})$$

The function E_1 is quasi-periodic, $E_1(z+1) = E_1(z)$, $E_1(z+\tau) = E_1(z) - 2\pi i$, while E_2 is double-periodic: $E_2(z+1) = E_2(z)$, $E_2(z+\tau) = E_2(z)$. Near $z=0$ they have the expansions

$$E_1(z) = \frac{1}{z} - 2\eta z + \dots, \quad E_2(z) = \frac{1}{z^2} + 2\eta + \dots$$

It is not difficult to see that the function $E_1(z)$ has the following values at the half-periods:

$$E_1(\omega_j) = -2\pi i \partial_\tau \omega_j \quad (\text{B18})$$

and, therefore, the identity

$$E_1(\omega_j) + E_1(\omega_k) = E_1(\omega_j + \omega_k) \quad (\text{B19})$$

holds true for any different $j, k = 1, 2, 3$.

The following function appears to be useful in the calculations:

$$\Phi(u, z) = \frac{\vartheta_1(u+z)\vartheta'_1(0)}{\vartheta_1(u)\vartheta_1(z)}. \quad (\text{B20})$$

It obeys the obvious properties $\Phi(u, z) = \Phi(z, u)$, $\Phi(-u, -z) = -\Phi(u, z)$ as well as less obvious ones:

$$\Phi(u, z)\Phi(-u, z) = \wp(z) - \wp(u) \quad (\text{B21})$$

$$\Phi(u, z)\Phi(w, z) = \Phi(u + w, z)(E_1(z) + E_1(u) + E_1(w) - E_1(z + u + w)). \quad (\text{B22})$$

Here $E_1(z)$ is the first Eisenstein function. The expansion of the function $\Phi(u, z)$ near $z = 0$ is

$$\Phi(u, z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + O(z^2). \quad (\text{B23})$$

The quasi-periodicity properties of the function Φ are:

$$\Phi(u, z + 1) = \Phi(u, z), \quad \Phi(u, z + \tau) = e^{-2\pi i u} \Phi(u, z). \quad (\text{B24})$$

The z -derivative of the function Φ is equal to

$$\partial_z \Phi(u, z) = \Phi(u, z)(E_1(u + z) - E_1(z)). \quad (\text{B25})$$

Finally, let us introduce the functions

$$\varphi_j(z) = e^{2\pi i z \partial_\tau \omega_j} \Phi(z, \omega_j), \quad j = 1, 2, 3. \quad (\text{B26})$$

Setting u in (B21) and (B22) to be equal to the half-periods, we have:

$$\varphi_j^2(z) = \wp(z) - e_j, \quad \varphi_j^2(z) - \varphi_k^2(z) = e_k - e_j \quad (\text{B27})$$

$$\varphi_j(z)\varphi_k(z) = \varphi_l(z)(E_1(z) + E_1(\omega_l) - E_1(z + \omega_l)). \quad (\text{B28})$$

In a similar way, from (B25) and (B18) it follows that

$$\partial_z \varphi_j(z) = \varphi_j(z) [E_1(z + \omega_j) - E_1(\omega_j) - E_1(z)] = -\varphi_k(z)\varphi_l(z), \quad (\text{B29})$$

where j, k, l is any cyclic permutation of 1, 2, 3.

Heat equation and related formulae

As it can be easily seen from the definition (B1), all the theta-functions satisfy the “heat equation”

$$4\pi i \partial_\tau \vartheta_a(z|\tau) = \partial_z^2 \vartheta_a(z|\tau) \quad (\text{B30})$$

or, in terms of the variable $t = \frac{\tau}{2\pi i}$ used in the main text, $2\partial_t \vartheta_a(z) = \partial_z^2 \vartheta_a(z)$. One can also introduce the “heat coefficient” $\kappa = \frac{1}{2\pi i}$ and rewrite the heat equation in the form $\partial_\tau \vartheta_a(z|\tau) = \frac{\kappa}{2} \partial_z^2 \vartheta_a(z|\tau)$. All formulas for derivatives of elliptic functions with respect to the modular parameter are based on the heat equation.

The τ -derivatives of the functions Φ , E_1 and E_2 are given by the following proposition.

Proposition 1 *The identities*

$$\partial_\tau \Phi(z, u) = \kappa \partial_z \partial_u \Phi(z, u), \quad (\text{B31})$$

$$\partial_\tau E_1(z) = \frac{\kappa}{2} \partial_z (E_1^2(z) - \wp(z)), \quad (\text{B32})$$

$$\partial_\tau E_2(z) = \kappa E_1(z) E_2'(z) - \kappa E_2^2(z) + \frac{\kappa}{2} \wp''(z), \quad (\text{B33})$$

with the “heat coefficient” $\kappa = \frac{1}{2\pi i}$, hold true³.

³(B31) was obtained in [25],[35].

Proof: First we prove (B31). It follows from (B30) that

$$4\pi i \frac{\partial_\tau \vartheta_1(z)}{\vartheta_1(z)} = \frac{\vartheta_1''(z)}{\vartheta_1(z)} = \partial_z \left(\frac{\vartheta_1'(z)}{\vartheta_1(z)} \right) + \left(\frac{\vartheta_1'(z)}{\vartheta_1(z)} \right)^2 = -E_2(z) + E_1^2(z). \quad (\text{B34})$$

Therefore,

$$\partial_\tau \Phi(z, u) = \frac{\kappa}{2} \left(-6\eta - E_2(z+u) + E_1^2(z+u) + E_2(z) - E_1^2(z) + E_2(u) - E_1^2(u) \right), \quad (\text{B35})$$

where the constant η is given by (B8). On the other hand,

$$\begin{aligned} \partial_z \partial_u \Phi(z, u) &= \partial_z \left[\Phi(z, u) (E_1(z+u) - E_1(u)) \right] = \Phi(z, u) \\ &\times \left[(E_1(z+u) - E_1(u))(E_1(z+u) - E_1(z)) - E_2(z+u) \right]. \end{aligned} \quad (\text{B36})$$

The rest of the proof is a direct use of the identity

$$(E_1(z+u) - E_1(u) - E_1(z))^2 = \wp(z) + \wp(u) + \wp(z+u). \quad (\text{B37})$$

Equation (B32) easily follows from (B31) and the local expansion (B23) around $u = 0$. Equation (B33) is just a derivative of (B32). \blacksquare

Next let us prove (8.22)⁴.

Proposition 2 Set $X(z) = \frac{\wp(z) - e_1}{e_2 - e_1}$, then

$$\partial_\tau X = \kappa \partial_z X \partial_z \log \theta_0(z). \quad (\text{B38})$$

Proof: The τ -derivative of $X(z) = \frac{\wp(z) - e_1}{e_2 - e_1} \stackrel{(\text{B27})}{=} \frac{\varphi_1^2(z)}{e_2 - e_1}$ is:

$$\partial_\tau X = \frac{2\varphi_1(z) \partial_\tau \varphi_1(z) (e_2 - e_1) - \partial_\tau (e_2 - e_1) \varphi_1^2(z)}{(e_2 - e_1)^2}.$$

Using the definition of $\varphi_1(z)$ and the “heat equation” (B31) for the Φ -function, we write

$$\begin{aligned} \partial_\tau \varphi_1(z) &\stackrel{(\text{B25})}{=} \kappa \partial_z \left[\varphi_1(z) (E_1(z + \omega_1) - E_1(\omega_1)) \right] \\ &= \kappa \partial_z \left[\varphi_1(z) E_1(z + \omega_1) \right] \\ &= \kappa \partial_z \varphi_1(z) E_1(z + \omega_1) - \kappa \varphi_1(z) E_2(z + \omega_1). \end{aligned} \quad (\text{B39})$$

Substituting this and $\partial_\tau (e_2 - e_1) \stackrel{(\text{B13})}{=} -2\kappa(e_2 - e_1)E_2(e_3)$ into (B39), we have:

$$\partial_\tau X = \frac{2\kappa}{e_2 - e_1} \left(\varphi_1(z) \partial_z \varphi_1(z) E_1(z + \omega_1) - \varphi_1^2(z) E_2(z + \omega_1) + E_2(\omega_3) \varphi_1^2(z) \right). \quad (\text{B40})$$

⁴This formula was proved by K.Takasaki in [29] by comparison of analytic properties of the both sides. Here we give another proof by a direct computation.

Since $\partial_z X = \frac{2\varphi_1(z)\partial_z\varphi_1(z)}{e_2 - e_1}$, we can rewrite the latter equation as

$$\partial_\tau X = \kappa \partial_z X E_1(z + \omega_1) + \frac{2\kappa\varphi_1^2(z)}{e_2 - e_1}(-E_2(z + \omega_1) + E_2(\omega_3))$$

which can be further simplified with the help of the identity

$$E_2(z + \omega_1) = E_2(\omega_1) + \frac{(e_2 - e_1)(e_3 - e_1)}{\varphi_1^2(z)}.$$

Dividing both sides by $\partial_z X$, we get

$$\frac{\partial_\tau X}{\partial_z X} = \kappa E_1(z + \omega_1) + 2\kappa(e_3 - e_1) \frac{X - 1}{\partial_z X}. \quad (\text{B41})$$

The last term can be transformed using the identities $e_3 - e_1 \stackrel{(\text{B27})}{=} \varphi_1^2(z) - \varphi_3^2(z)$, $\partial_z X \stackrel{(\text{B29})}{=} -2\frac{\varphi_1(z)\varphi_2(z)\varphi_3(z)}{e_2 - e_1}$ and $X - 1 \stackrel{(\text{B27})}{=} \frac{\varphi_2^2(z)}{e_2 - e_1}$:

$$\partial_\tau X = \kappa \partial_z X \left(E_1(z + \omega_1) + \frac{\varphi_2(z)\varphi_3(z)}{\varphi_1(z)} - \frac{\varphi_1(z)\varphi_2(z)}{\varphi_3(z)} \right). \quad (\text{B42})$$

Finally, the desired formula (B38) is obtained from this using (B28):

$$\partial_\tau X = \kappa \partial_z X (E_1(z + \omega_3) - E_1(\omega_3)) = \kappa \partial_z X \partial_z \log \theta_0(z). \quad (\text{B43})$$

■

Appendix C

Gauge transformation of the linear problems for P_{VI}

In the parametrization (8.10), (8.12), the upper right entries of the matrices $\mathbf{U}(X, T)$, $\mathbf{V}(X, T)$ forming the modified Jimbo-Miwa U - V pair for the P_{VI} equation are

$$\mathbf{U}_{12} = \mathbf{b} = \frac{K(X - y)}{X(X - 1)(X - T)}, \quad \mathbf{V}_{12} = \mathbf{B} = \frac{K(y - T)}{T(T - 1)(X - T)}.$$

Passing to a parametrization $X = X(x, t)$, $T = T(t)$ according to the rule (2.27) and performing a diagonal gauge transformation of the form (2.9) we get the following expressions for the upper right entries of the matrices $\mathbf{U}(x, t)$, $\mathbf{V}(x, t)$:

$$b = \mathbf{U}_{12} = \mathbf{b} X_x \omega^2 = \frac{K(X - y)}{X(X - 1)(X - T)} X_x \omega^2, \quad (\text{C1})$$

$$B = \mathbf{V}_{12} = (T_t \mathbf{B} + X_t \mathbf{b}) \omega^2 = \frac{K(y - T)}{T(T - 1)(X - T)} T_t \omega^2 + \frac{K(X - y)}{X(X - 1)(X - T)} X_t \omega^2. \quad (\text{C2})$$

The x -derivative of b is

$$b_x = \frac{(X-y)\omega^2}{X(X-1)(X-T)} \left(f + \frac{X_x^2}{X-y} \right), \quad (\text{C3})$$

where the notation

$$f = X_{xx} + X_x \partial_x \log(\omega^2) - X_x^2 \left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-T} \right)$$

is introduced for brevity. Further, let us impose condition of the form (2.4):

$$b_x = kB, \quad (\text{C4})$$

with some constant k (not yet fixed). Substituting (C2) and (C3) into (C4), we obtain an equality of two linear functions of y *provided* $\partial_x \log \omega$ *does not depend on* y . (The latter assumption is necessary to achieve separation of the variables x, u in the non-stationary Schrödinger equation.) Assuming this, we equate the coefficients in front of y and the y -independent terms in the both sides and get the system of two equations

$$\begin{cases} f = k \left(X_t - \frac{X(X-1)}{T(T-1)} T_t \right) \\ Xf + (X_x)^2 = kX \left(X_t - \frac{X-1}{T-1} T_t \right). \end{cases} \quad (\text{C5})$$

from which the functions $X(x, t)$ and $\partial_x \log \omega$ can be determined. Excluding f , we arrive at the differential equation for X :

$$X_x^2 = \frac{kT_t}{T(T-1)} X(X-1)(X-T). \quad (\text{C6})$$

We know that T_t is given by (8.23): $T_t = 2(e_2 - e_1)T(T-1)$. Therefore,

$$X_x^2 = 2k(e_2 - e_1)X(X-1)(X-T). \quad (\text{C7})$$

This relation prompts the elliptic parametrization (8.16) and fixes the value of k :

$$k = 2. \quad (\text{C8})$$

Note that in some sense this is “the same” coefficient 2 that enters the heat equation for theta-functions in the t -variable $t = \kappa\tau$: $2\partial_t \vartheta_a(x) = \partial_x^2 \vartheta_a(x)$. In the same sense the non-stationary Schrödinger equation for the ψ -function is a “dressed” version of the heat equation.

Now we are ready to fix the x -dependent part of the function ω^2 . From the first equation of the system (C5) we find:

$$\partial_x \log \omega^2 = -\frac{X_{xx}}{X_x} + X_x \left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-T} \right) - k \frac{X(X-1)}{T(T-1)} \frac{T_t}{X_x} + k \frac{X_t}{X_x}. \quad (\text{C9})$$

It is easy to show that $\frac{X_{xx}}{X_x} = \frac{X_x}{2} \left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-T} \right)$, so plugging the previously obtained formulas for X_t and T_t into (C9), we get:

$$\partial_x \log \omega^2 = \frac{X_x}{2} \left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-T} \right) - 4(e_2 - e_1) \frac{X(X-1)}{X_x} + 2E_1(x + \omega_3) - 2E_1(\omega_3). \quad (\text{C10})$$

To proceed, we substitute

$$X = \frac{\varphi_1^2(x)}{e_2 - e_1}, \quad X - 1 = \frac{\varphi_2^2(x)}{e_2 - e_1}, \quad X - T = \frac{\varphi_3^2(x)}{e_2 - e_1}$$

and

$$X_x = -2 \frac{\varphi_1(x)\varphi_2(x)\varphi_3(x)}{e_2 - e_1}.$$

This yields

$$\partial_x \log \omega^2 = -\frac{\varphi_2(z)\varphi_3(z)}{\varphi_1(z)} - \frac{\varphi_1(z)\varphi_3(z)}{\varphi_2(z)} + \frac{\varphi_1(z)\varphi_2(z)}{\varphi_3(z)} + 2E_1(x + \omega_3) - 2E_1(\omega_3). \quad (\text{C11})$$

The final result obtained with the help of (B28) is

$$\partial_x \log \omega^2 = -E_1(x) + \sum_{j=1}^3 [E_1(x + \omega_j) - E_1(\omega_j)], \quad (\text{C12})$$

or, in the integrated form,

$$\omega^2(x, t) = \frac{\vartheta_2(x)\vartheta_3(x)\vartheta_0(x)}{\vartheta_1(x)} g(y, t), \quad (\text{C13})$$

where the function $g(y, t)$ can not be fixed by the above arguments. Using the identity

$$2 \frac{\vartheta'_1(0)\vartheta_0(0)}{\vartheta_2(0)\vartheta_3(0)} \frac{\vartheta_2(x)\vartheta_3(x)}{\vartheta_1(x)\vartheta_0(x)} = -\frac{\wp'(x)}{\wp(x) - e_3},$$

we can express ω^2 in terms of the \wp -function:

$$\omega^2(x, t) = \frac{\wp'(x)\theta_0^2(x)}{2(\wp(x) - e_3)} \rho^2(t) \quad (\text{C14})$$

with some $\rho(t)$ to be determined from the condition that the x -independent part of the potential in the non-stationary Schrödinger equation be equal to the classical Hamiltonian $H_{\text{VI}}(\dot{u}, u)$. It is the form (C14) that is more convenient to use in Section 8.3.

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